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# QUASI-SPHERICAL METRICS AND PRESCRIBED SCALAR CURVATURE

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**Abstract**

We describe a construction for metrics of prescribed scalar curvature on $S^2 \times \mathbf{R}$, based on a *quasi-spherical* coordinate condition. The construction uses two arbitrary functions and requires the solution of a semilinear parabolic equation on $S^2$, with the arbitrary functions and the scalar curvature appearing as source terms. We obtain existence results for this equation under various geometrically natural boundary conditions, and thereby construct some 3-metrics of interest in general relativity.

## 1. Introduction

Riemannian 3-manifolds with prescribed scalar curvature arise naturally in general relativity as spacelike hypersurfaces in the underlying spacetime. If $g = (g_{ij})$, $i, j = 1, \cdots, 3$, is the induced (Riemannian) metric on the spacelike hypersurface $M$, then the scalar curvature $R(g)$ is determined by the extrinsic curvature (second fundamental form) $K_{ij}$ and the spacetime energy-momentum tensor $T_{\alpha\beta}$, via the Gauss-Codazzi and Einstein equations:

$$16\pi T(e_0, e_0) = R(g) - \|K\|^2 + (\operatorname{tr}_g K)^2, \tag{1.1}$$

where $e_0$ is the (future) timelike unit normal of the hypersurface $M$, $\|K\|^2 = g^{ik}g^{jl}K_{ij}K_{kl}$, $\operatorname{tr}_g K = g^{ij}K_{ij}$, and the Einstein equations are $G_{\alpha\beta} := \operatorname{Ric}_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}$. The main situation of physical interest is where $R(g) \geq 0$—for example, if $M$ is totally geodesic $(K_{ij} = 0)$ and the spacetime is vacuum $(T_{\alpha\beta} = 0)$, then $R(g) = 0$, and more generally if $M$ is a *maximal* hypersurface $(\operatorname{tr}_g K = 0)$ and the spacetime satisfies the *weak energy condition* [18], then $T(e_0, e_0) \geq 0$ and thus $R(g) \geq 0$. Provided $M$ is suitably constrained (for example, by the maximal hypersurface condition), the metric structure of $(M, g)$ reflects that of the ambient spacetime, and therefore it is important to understand this structure.

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In the present paper we describe a new construction for 3-metrics of prescribed scalar curvature, based on the assumption of a foliation by constant Gauss curvature 2-spheres. We term such a foliation *quasi-spherical* (QS). Assuming further that the radius function $r$ of the foliation is a smooth coordinate, the metric can be written in the form

$$g = u^2 dr^2 + (\beta_1 dr + r\, d\vartheta)^2 + (\beta_2\, dr + r \sin\vartheta\, d\varphi)^2, \tag{1.2}$$

where $u$ and $\beta_A$, $A = 1, 2$, are unspecified metric components. The significance of this coordinate condition stems from the surprising fact that the equation for the scalar curvature $R(g)$ can be rewritten as a semilinear *parabolic* equation (see (3.3)) for $u$, using the standard Laplacian on $S^2$ and with $\log r$ playing the role of "time." The functions $R(g)$ and $\beta_A$, $A = 1, 2$, then appear in source terms for the parabolic equation.

The major part of this paper is devoted to establishing properties of the parabolic scalar curvature equation, beginning by determining explicit size conditions on the source functions which ensure the solution $u$ is strictly positive and regular. The size conditions turn out to be mild (see Theorem 3.7), thereby giving a large family of metrics (1.2) with prescribed scalar curvature. Here, and throughout this paper, by "prescribed" we mean $R(g)(r, \vartheta, \varphi) = R_M(r, \vartheta, \varphi)$, where $R_M \in C^\infty(\mathbf{R}^3)$ is given and $(r, \vartheta, \varphi)$ are identified with the standard polar coordinates on $\mathbf{R}^3$. Although the motivating problems concern nonnegative scalar curvature functions, the construction works equally well (if not better) with negative prescribed scalar curvature.

We consider three types of initial condition for $u$, corresponding to the geometric conditions for regularity across $r = 0$ (Theorem 4.3), minimal surface boundary at $r = r_0 > 0$ (Theorem 4.6), and prescribed (positive) mean curvature (Theorem 3.7, Corollary 3.6). We also describe natural decay conditions for $R_M$ and $\beta_A$ which ensure the metric is asymptotically flat, in the sense required for the positive mass theorem ([26], [31]) (Theorem 4.2). The existence results for complete asymptotically flat solutions are collected in Theorem 4.5, and stated in terms of rectangular rather than spherical polar coordinates on $\mathbf{R}^3$.

By choosing appropriate $R_M$ and $\beta_A$, particular solutions with interesting properties can be constructed. For example, requiring $\beta_A = 0 = R_M$ for $r \leq 1$ (and $\beta_A$, $R_M$ otherwise free, subject only to the size constraints of Theorem 3.7), we obtain a family of metrics on $\mathbf{R}^3$, each containing a region isometric with a flat ball (Corollary 4.4). If in addition we choose $R_M = 0$ for all $r$, then this metric gives totally geodesic initial data for the vacuum Einstein equations, and the resulting maximally extended spacetime [8] is in general not flat, but contains a region which is isometric to a region in flat Minkowski space.

A similar idea but with black hole (minimal surface) boundary conditions, leads to a family of metrics with interior region isometric to parts of the classical Schwarzschild metric. Recall that the Schwarzschild 3-metric is

$$ds^2_{\text{Schw}} = \frac{dr^2}{1 - 2M/r} + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

for $r > 2M$, where $M$ is a positive constant, and $r = 2M$ is a totally geodesic $S^2$ boundary (representing the intersection of the past and future event horizons in the Schwarzschild spacetime [18]). Choosing $\beta_A = 0 = R_M$ for $2M \le r \le 2M + 1$ (say), and imposing the singular initial condition $u^{-1}(r = 2M) = 0$ leads to metrics isometric to $ds^2_{\text{Schw}}$ for $2M \le r \le 2M + 1$.

An interesting conjecture of Penrose ([24], [19], [16]) proposes that $ds^2_{\text{Schw}}$ has the least total (ADM) mass ([1], [3]) from among all 3-metrics of nonnegative scalar curvature and having minimal surface boundary of a fixed area. This may be considered a generalization of the positive mass theorem, which proves that $\mathbf{R}^3$ is similarly distinguished from among *complete* 3-metrics of nonnegative scalar curvature. Support for the Penrose conjecture is provided by the class of QS metrics having *divergence-free shear*,

$$\operatorname{div}\beta = \nabla_A\beta_A = 0, \tag{1.3}$$

where $\nabla_A$ is the covariant derivative on $S^2$. The divergence-free condition reduces the freedom in $\beta_A$ to one function of three variables, as can be seen from the Helmholtz-Hodge decomposition on $S^2$,

$$\beta_A = \nabla_A f_1 + \varepsilon_{AB}\nabla_B f_2, \tag{1.4}$$

for some functions $f_1$, $f_2$. Now $\operatorname{div}\beta = 0$ is equivalent to $f_1 = \text{const}$, hence $\beta_A$ is determined just by $f_2$.

Defining

$$M(r) = \frac{1}{8\pi}\oint_{S_r} r(1 - u^{-2})\, d\sigma, \tag{1.5}$$

we show under appropriate decay conditions on $\beta_A$ and $R_M$ (Theorem 4.2) that the total mass is given by $m_{\text{ADM}} = \lim_{r\to\infty} M(r)$, and an easy calculation using (3.3) and the divergence-free condition (1.3) shows

$$\frac{d}{dr}M(r) = \frac{1}{8\pi}\oint_{S_r}\left[u^{-2}\left(|\nabla u|^2 + \frac{1}{2}|\beta_{(A|B)}|^2\right) + \frac{1}{2}R_M r^2\right]. \tag{1.6}$$

Note that $M(r)$ agrees with the Hawking mass [10] when $\operatorname{div}\beta = 0$, but not in general. Imposing the initial condition $u^{-1}(r_0) = 0$ and assuming $R_M \geq 0$ we have $m_{\text{ADM}} \geq \frac{1}{2}r_0$, which is exactly the Penrose inequality. This shows the Penrose conjecture holds for the class of QS metrics with divergence-free shear and having interior boundary totally geodesic and isometric to $r_0^2 S^2$. More formally, we have

**Corollary 1.1.** *Suppose* $\beta_A$ *and* $R_M$ *satisfy the conditions of Theorems* 3.7, 4.2, *and* 4.6 *on* $A_{[r_0;\infty)} = S^2 \times [r_0;\infty)$, *and, in addition, suppose*

$$R_M \geq 0, \qquad \operatorname{div}\beta = 0, \tag{1.7}$$

*and let* $u$ *be the solution of* (3.3) *with initial condition* $u^{-1}(r_0) = 0$. *Then the metric* (1.2) *has totally geodesic boundary at* $r = r_0$ *and total* $(ADM)$ *mass*

$$m_{\text{ADM}} \geq \tfrac{1}{2}r_0.$$

An important motivational application concerns the extension problem, which was suggested by the definition of quasi-local mass in [4]. This problem asks:

> given a bounded Riemannian 3-manifold $(\Omega, g_0)$, describe the class of complete 3-manifolds $(M, g)$ satisfying the conditions of the positive mass theorem (in particular, asymptotically flat with nonnegative scalar curvature) and containing $(\Omega, g_0)$ isometrically.

If we consider this as a problem of matching $M\backslash\Omega$ with $\Omega$ across $\Sigma = \partial(M\backslash\Omega)$, then the condition that the scalar curvature be defined distributionally and bounded across $\Sigma$ leads to the geometric boundary conditions

$$g|_{T\Sigma} = g_0|_{T\partial\Omega}, \qquad H_{\Sigma,g} = H_{\partial\Omega,g_0}, \tag{1.8}$$

where $H_{\Sigma,g}$ is the mean curvature of $\Sigma$ in $(M, g)$, and the unit normals of $\partial\Omega$ and $\Sigma$ are chosen oriented consistently, with the normalization giving a sphere of radius $r$ in $\mathbf{R}^3$ mean curvature $+2/r$. The condition that the full curvature tensor be bounded is more restrictive, implying the boundary condition for the full second fundamental form,

$$\mathrm{II}_{\Sigma,g} = \mathrm{II}_{\partial\Omega,g_0}. \tag{1.9}$$

We will not consider this boundary condition in the present paper. Since the mean curvature condition translates into a Dirichlet condition for $u$ (2.17), we see that if $(\partial\Omega, g_0) = (S^2, r_0^2\,d\sigma^2)$ and $H_{\partial\Omega,g_0} > 0$, the QS technique provides a large class of metrics extending $(\partial\Omega, g_0)$. We note that it is not possible to construct such extensions using the traditional conformal method [8], since (1.8) generates incompatible boundary conditions for the conformal factor.

The Penrose conjecture argument above applies equally well to the extension problem, for the particular case where $(\partial\Omega, g_0) = (S^2, r_0^2\, d\sigma^2)$ and $H_{\partial\Omega, g_0} = \text{const} > 0$. We thereby show that the Schwarzschild extension has the least total mass from among all quasi-spherical divergence-free shear extensions of nonnegative scalar curvature, which satisfy the mean curvature boundary conditions (1.8). This is in accordance with the *static metric conjecture* of [4], which conjectures in general that the minimum mass extension is achieved by a metric satisfying the spacetime static metric equations.

In the final section, we analyze the behavior of the QS metric as $r \to \infty$, for the special case $\beta_A, R_M \equiv 0$ for $r \geq r_0$, and describe carefully the decay to the Schwarzschild metric (Theorem 5.1). As well as illustrating the spherical harmonic decomposition technique, this result should be useful in the numerical construction of initial data metrics.

The shear vector $\beta_A$ gives two functions of three variables to describe a 3-metric with prescribed scalar curvature, and on heuristic grounds one might expect that this parametrization covers an open set of such metrics (in the space of all smooth metrics, for example). In future work we will show that this expectation is justified, by showing that the set of metrics admitting a QS foliation contains an open set in the space of smooth metrics, and that the local QS gauge freedom is determined by six functions of one variable only [6].

The primary motivation for this investigation was the extension problem in the class of positive-mass metrics, which in turn arose from the definition of quasi-local mass [4]. The idea of using a foliation by *metric* 2-spheres was suggested by work of P. Szekeres [29], who described a class of dust spacetimes, generalizing the (spherically symmetric) Tolman-Bondi spacetimes. The Szekeres spacetime metrics admit a foliation by metric 2-spheres (and the term *quasi-spherical* is due to him), but the metric form in [29] uses coordinates which do not emphasize the quasi-spherical structure, and the shear vector is restricted to a 5-dimensional family of vector fields on $S^2$.

The idea of using a (topological) 2-sphere foliation to describe the dynamics of the Einstein equations is well known, occurring first in the classical work of Bondi and Sachs ([7], [25]), and more recently in the detailed and pioneering analysis of the global small data existence question for the Einstein equations by Christodoulou and Klainerman [9]. In these works the foliation is determined by the affine distance function along null geodesics generating a foliation by null 3-surfaces.

Geometrically-based foliations of 3-dimensional Riemannian manifolds (space-like hypersurfaces) have been used in various attempts to prove the positive-mass theorem. Geroch ([15], [19]) showed that a global 2-sphere foliation satisfying the (parabolic) heat flow by inverse mean curvature leads to a proof of the positive-mass theorem, and also to the Penrose conjecture. However, existence results for this flow have only recently been shown in flat $\mathbf{R}^3$ ([14], [30]), and it seems difficult to generalize these to nonflat metrics. Kijowski [20] showed that a foliation defined by level sets of a solution of a $p$-harmonic equation also leads to the positive-mass theorem, and existence results were obtained by Chruściel [11]. Again, it is unlikely the level sets will form a smooth foliation in general metrics.

The above applications of foliations are all *descriptive*—starting with a space-time (or space-like hypersurface), a foliation is imposed, in order to better describe the metric. The approach taken in this paper is instead *constructive*, and is most commonly considered using the conformal method [8]. However, there are some problems for which the QS technique is more suitable than the conformal method. For example, as has already been indicated, the conformal method is not compatible with the geometric boundary conditions, and thus cannot be used to construct extension metrics. In numerical relativity, the elliptic equations for the conformal factor are expensive to solve [23], and it is interesting to note that a coordinate-based parabolic construction has been suggested, in order to sidestep this difficulty ([2], [28]). Although there does not appear to be a natural geometric description of the foliation used in [2], the related "polar" time gauge is also closely related to the quasi-spherical foliation condition [5].

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## 2. Curvature calculations

Let $(M, g)$ be a Riemannian 3-manifold, with foliation function $r \in C^\infty(M)$. This means $dr \neq 0$ and the level sets

$$S_r = \{p \in M : r(p) = r\} \tag{2.1}$$

form a $C^\infty$ foliation of $M$. We say $r$ determines a *quasi-spherical* foliation if $r$ is positive and

$$(S_r, g|_{S_r}) \cong (S^2, r^2 d\sigma^2), \tag{2.2}$$

where $d\sigma^2 = \sigma_1^2 + \sigma_2^2 = \sum \sigma_A^2$ is the standard metric on the unit 2-sphere $S^2$. We construct quasi-spherical coordinates

$$(r, \vartheta) \colon M \to I \times S^2, \qquad I \subset \mathbf{R}^+, \tag{2.3}$$

as follows. Choose any $C^\infty$ curve $r \mapsto c(r)$ transverse to the leaves $S_r$ and any unit vector field $r \mapsto v(r)$ along $c(r)$ such that $v(r)$ is tangent to $S_r$, and use $c(r)$ and $v(r)$ to determine an isometry (up to scaling) $S_r \overset{\cong}{\to} S^2$—for example, use $c(r)$ to fix the South Pole ($\vartheta = \pi$ in polar coordinates) and $v(r)$ to determine the Greenwich Meridian ($\varphi = 0$ in polar coordinates). This defines a projection

$$\pi \colon M \to S^2, \tag{2.4}$$

and angular coordinates $\pi^*(\vartheta)$, $\vartheta \in S^2$, on $M$ such that

$$g|_{TS_r} = r^2 \pi^*(d\sigma^2)|_{TS_r}. \tag{2.5}$$

Following the standard abuse of notation we write $\vartheta$ for $\pi^*(\vartheta)$, the angular coordinates on $M$, and $\sigma_A$ for $\pi^*(\sigma_A)$, the angular 1-forms on $M$.

**Lemma 2.1.** *There are functions $u(r, \vartheta)$ and $\beta_A(r, \vartheta)$, $A = 1, 2$, such that the metric $g$ in the QS coordinates $(r, \vartheta)$ on $M$ determined by the QS projection $\pi \colon M \to S^2$ is*

$$g = u^2 dr^2 + \sum_{A=1}^{2} (\beta_A dr + r\sigma_A)^2. \tag{2.6}$$

*The functions $u$ and $\beta_A$ are described invariantly by*

$$u^{-2} = g(\widehat{\nabla} r, \widehat{\nabla} r), \qquad \beta_A = -ru^2 \sigma_A(\widehat{\nabla} r), \tag{2.7}$$

*where $\widehat{\nabla}$ is the covariant derivative of $g$, and $\widehat{\nabla} r$ is the gradient vector.*

*Proof.* Since $(dr, \sigma_A)$ are linearly independent in $T^*M$, from (2.5) we have $g = r^2 d\sigma^2 + dr \cdot (\textit{something})$, and the form (2.6) arises by expanding $\textit{something} = 2r\beta_A\sigma_A + (u^2 + \beta^2)\, dr$, where $\beta^2 = \beta_1^2 + \beta_2^2$ (note $u^2 > 0$ since $g$ is Riemannian). The formulae (2.7) follow from

$$\widehat{\nabla} r = u^{-2}(\partial_r - r^{-1}\beta_A v_A), \tag{2.8}$$

where $v_A \in TS_r$, $A = 1, 2$, is the frame dual to $\{\sigma_A\}$. q.e.d.

The coframe

$$\theta_A = \beta_A\, dr + r\sigma_A, \quad A = 1, 2, \qquad \theta_3 = u\, dr, \tag{2.9}$$

satisfies $g = \theta_1^2 + \theta_2^2 + \theta_3^2$, and has dual frame

$$e_A = r^{-1} v_A, \quad A = 1, 2, \qquad e_3 = u^{-1}(\partial_r - r^{-1}\beta_A v_A), \tag{2.10}$$

where $\{v_A\}$ is a frame tangent to $S_r$ and dual to $\{\sigma_A\}$, hence $e_A$ are tangent to $S_r$.

Since the calculations to follow are most naturally expressed in terms of the geometry of the standard $S^2$, we adopt the conventions that tensor indices $A, B, \cdots$ refer to the vectors $\{v_A\}$, while indices $\widehat{A}, \widehat{B}, \cdots$ refer to the $g$-orthonormal vectors $e_A = r^{-1} v_A$; so, for example

$$K_{\widehat{A}\widehat{B}} = K(e_A, e_B) = r^{-2} K_{AB} = r^{-2} K(v_A, v_B).$$

To avoid ambiguity about the metric used to raise indices, and thereby emphasize that the indices $A, B, \cdots$ refer to the $S^2$-orthonormal frame $v_1, v_2$, all indices will be written lowered. Geometrically, the calculation of the curvature of $g$ will be expressed in terms of the product metric $d\sigma^2 + dr^2$ and derivatives $r\partial_r$ and $\nabla$ on $S^2 \times \mathbf{R}^+$. For some purposes it is also useful to present calculations in terms of rectangular coordinates on $\mathbf{R}^3$, as described in §4.

We denote the connection of the metric $g$ by $\widehat{\nabla}$ and the connection matrix by $\omega_{ij} = g(e_i, \widehat{\nabla} e_j)$, $i, j = 1, \cdots, 3$. The connection of the metric $d\sigma^2$ on the level set $S_r$ is denoted $\nabla$, with connection matrix $\tau_{AB}$,

$$\tau_{AB} = \sigma_A(\nabla v_B) = d\sigma^2(v_A, \nabla v_B).$$

The structure equations of $d\sigma^2$ (pulled back by $\pi$ from $S^2$) are

$$d\sigma_A = -\tau_{AB}\sigma_B, \qquad \mathcal{T}_{AB} = d\tau_{AB} + \tau_{AC}\tau_{CB}, \tag{2.11}$$

where

$$\mathcal{T}_{AB} = -\tfrac{1}{2}\mathcal{T}_{ABCD}\sigma_C\sigma_D = r^{-2}\mathcal{T}_{ABCD}(-\tfrac{1}{2}\theta_C\theta_D + u^{-1}\beta_D\theta_C\theta_3), \tag{2.12}$$

and $\mathcal{T}_{ABCD} = \delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}$ is the curvature tensor of $S^2$. Denoting the index covariant derivative of $d\sigma^2$ by $\nabla_A \cdot = \cdot_{|A}$, the Ricci identity becomes $(\beta = \beta_A v_A)$

$$\begin{aligned} \beta_{A|BC} - \beta_{A|CB} &= d\sigma^2([\nabla_{v_C}, \nabla_{v_B}]\beta, v_A) \\ &= \mathcal{T}_{BCAD}\beta_D = \delta_{AC}\beta_B - \delta_{AB}\beta_C. \end{aligned} \tag{2.13}$$

The symmetrization $\beta_{(A|B)}$ and antisymmetrization $\beta_{[A|B]}$ of $\beta_{A|B}$ are defined by

$$\beta_{(A|B)} = \tfrac{1}{2}(\beta_{A|B} + \beta_{B|A}), \qquad \beta_{[A|B]} = \tfrac{1}{2}(\beta_{A|B} - \beta_{B|A}),$$

and we note the following consequence of the Ricci identity:

$$\beta_{(A|B)C} - \beta_{(A|C)B} = \beta_{[B|C]A} + \mathscr{T}_{BCAD}\beta_D. \tag{2.14}$$

It is now readily verified that

$$\begin{aligned} d\theta_A &= -\tau_{AB}\theta_B - r^{-1}u^{-1}(\theta_A - \beta_{A|B}\theta_B)\theta_3, \\ d\theta_3 &= -r^{-1}u^{-1}u_{|A}\theta_3\theta_A, \end{aligned}$$

where we use the formula

$$d\beta_A = \beta_{A|B}\sigma_B - \beta_B\tau_{AB} + \partial_r(\beta_A)\,dr. \tag{2.15}$$

The connection 1-form $\omega_{ij}$ has components

$$\begin{aligned} \omega_{\widehat{A}\widehat{B}} &= \tau_{AB} + r^{-1}u^{-1}\beta_{[A|B]}\theta_3, \\ \omega_{\widehat{A}3} &= r^{-1}u^{-1}((\delta_{AB} - \beta_{(A|B)})\theta_B - u_{|A}\theta_3), \end{aligned} \tag{2.16}$$

where

$$\tau_{AB} = d\sigma^2(v_A, \nabla v_B) = \tau_{ABC}\sigma_C = \tau_{ABC}\cdot r^{-1}(\theta_A - u^{-1}\beta_A\theta_3).$$

From (2.16) the second fundamental form $\mathrm{II}_{AB}$ and mean curvature $H$ of the surface $S_r$ are given by

$$\mathrm{II}_{AB} = -g(\widehat{\nabla}_{e_A}e_B, e_3) = r^{-1}u^{-1}(\delta_{AB} - \beta_{(A|B)}), \tag{2.17}$$

$$H = r^{-1}u^{-1}(2 - \operatorname{div}\beta), \tag{2.18}$$

where the normalization sets $H = +2/r$ for a sphere of radius $r$ in $\mathbf{R}^3$. The curvature 2-forms $\Omega_{\widehat{A}\widehat{B}}$ and $\Omega_{\widehat{A}3}$ are now found to be

(2.19)

$$\begin{aligned} \Omega_{\widehat{A}\widehat{B}} &= r^{-2}u^{-2}[u\beta_{[A|B]C} + (\delta_{AC} - \beta_{(A|C)})u_{|B} - (\delta_{BC} - \beta_{(B|C)})u_{|A}]\theta_C\theta_3 \\ &\quad - r^{-2}u^{-2}(\delta_{AC} - \beta_{(A|C)})(\delta_{BD} - \beta_{(B|D)})\theta_C\theta_D + \mathscr{T}_{AB} \\ \Omega_{\widehat{A}3} &= r^{-2}u^{-2}[u\beta_{(A|B)C} + (\delta_{AB} - \beta_{(A|B)})u_{|C}]\theta_B\theta_C \\ &\quad + r^{-2}u^{-2}[(\delta_{AB} - \beta_{A|B})u^{-1}(r\partial_r u - \beta_C u_{|C}) - uu_{|AB} + r\partial_r\beta_{(A|B)} \\ &\quad - \beta_C\beta_{(A|B)C} + \beta_{(A|B)} + \beta_{[A|C]}\beta_{[B|C]} - \beta_{C|A}\beta_{C|B}]\theta_B\theta_3, \end{aligned}$$

from which the curvature tensor[1] $R_{ijkl}$ is

$$R_{\widehat{A}\widehat{B}\widehat{C}\widehat{D}} = r^{-2}\mathscr{T}_{ABCD} + r^{-2}u^{-2}[(\delta_{AC} - \beta_{(A|C)})(\delta_{BD} - \beta_{(B|D)}) - (\delta_{AD} - \beta_{(A|D)})(\delta_{BC} - \beta_{(B|C)})], \tag{2.20}$$

$$R_{\widehat{A}\widehat{B}\widehat{C}3} = r^{-2}u^{-2}[(\beta_{[A|B]C} + \mathscr{T}_{ABCD}\beta_D)u + (\delta_{AC} - \beta_{(A|C)})u_{|B} - (\delta_{BC} - \beta_{(B|C)})u_{|A}], \tag{2.21}$$

$$R_{\widehat{A}3\widehat{B}3} = -r^{-2}u^{-2}[(\delta_{AB} - \beta_{(A|B)})u^{-1}(r\partial_r u - \beta_C u_{|C}) - uu_{|AB} + r\partial_r\beta_{(A|B)} - \beta_C\beta_{(A|B)C} + \beta_{(A|B)} + \beta_{[A|C]}\beta_{[B|C]} - \beta_{C|A}\beta_{C|B}]. \tag{2.22}$$

Observe these calculations are valid more generally for an $n$-dimensional manifold $M^n$ with foliation leaves $(S_r, g_{|TS_r}) \cong (\Sigma^{n-1}, r^2 d\sigma^2)$ for $(\Sigma^{n-1}, d\sigma^2)$ any closed $(n-1)$-manifold with metric $d\sigma^2$ and curvature $\mathscr{T}_{AB}$. Metrics of this general form might be called *quasi-homothetic*, since the radial vector $\partial_r$ generalizes the usual homothety of $\mathbf{R}^3$. With only minor modifications, the following expressions for the Ricci tensor $\mathrm{Ric}_{ij} = g^{kl}R_{iklj}$ and scalar curvature $R_M = g^{ij}\mathrm{Ric}_{ij}$ also generalize. Using the form (2.12) of $\mathscr{T}_{AB}$, we have

$$\mathrm{Ric}_{\widehat{A}\widehat{B}} = r^{-2}u^{-2}[(\delta_{AB} - \beta_{(A|B)})u^{-1}(r\partial_r u - \beta_C u_{|C}) + \delta_{AB}(u^2 - 1) - uu_{|AB} + r\partial_r\beta_{(A|B)} - \beta_C\beta_{(A|B)C} + \beta_{(A|B)} + (\delta_{AB} - \beta_{(A|B)})\,\mathrm{div}\,\beta + \beta_{[A|C]}\beta_{(B|C)} + \beta_{(A|C)}\beta_{[B|C]}], \tag{2.23}$$

$$\mathrm{Ric}_{\widehat{A}3} = r^{-2}u^{-2}[(1 - \mathrm{div}\,\beta)u_{|A} + \beta_{(A|B)}u_{|B} - u(\beta_{[A|B]B} + \beta_A)], \tag{2.24}$$

$$\mathrm{Ric}_{33} = r^{-2}u^{-2}[(2 - \mathrm{div}\,\beta)u^{-1}(r\partial_r u - \beta_C u_{|C}) - u\Delta u + r\partial_r(\mathrm{div}\,\beta) - \beta_C(\mathrm{div}\,\beta)_{|C} + \mathrm{div}\,\beta - |\beta_{(A|B)}|^2], \tag{2.25}$$

where $\mathrm{div}\,\beta = \beta_{A|A}$, $|\beta_{(A|B)}|^2 = (\beta_{1|1})^2 + 2(\beta_{(1|2)})^2 + (\beta_{2|2})^2$, and the Ricci scalar $R(g) = R_M$ is given by

$$R_M = 2r^{-2}u^{-2}[(2 - \mathrm{div}\,\beta)u^{-1}(r\partial_r u - \beta_C u_{|C}) - u\Delta u + u^2 - 1 + r\partial_r(\mathrm{div}\,\beta) - \beta_C(\mathrm{div}\,\beta)_{|C} + 2\,\mathrm{div}\,\beta - \tfrac{1}{2}(\mathrm{div}\,\beta)^2 - \tfrac{1}{2}|\beta_{(A|B)}|^2]. \tag{2.26}$$

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[1] Our index convention for $R_{ijkl}$ sets $\Omega_{ij} = -\frac{1}{2}R_{ijkl}\theta_k\theta_l$.

We now make the basic observation that, by viewing $u$ as an unknown function and $R_M$ and $\beta_A$ as prescribed fields, this gives a parabolic partial differential equation for $u$ on $\mathbf{R}^+ \times S^2$. In the following sections we will study the solvability of this parabolic equation and the properties of the resulting QS metrics.

## 3. Existence for prescribed scalar curvature

The parabolic form of the scalar curvature equation (2.26) indicates that metrics with prescribed scalar curvature could be constructed by specifying the scalar curvature function $R_M(r, \vartheta)$, shear vector $\beta_A(r, \vartheta)$, and initial condition $u(r = r_0) = u_0$, and then solving (3.3) for $u$. In this section we describe conditions on $\beta_A$ and $R_M$ under which global existence for the initial value problem for $u$ can be shown (Theorem 3.4). This implies solvability of the geometric boundary problem (1.8) for prescribed strictly positive mean curvature and boundary isometric to $r_0^2 S^2$ (Corollary 3.6). Size conditions on $\beta_A$ and $R_M$ and the initial condition $u_0 \in C^{2,\alpha}(S^2)$ which ensure blowup for solutions of (3.3) in finite "time" are given in Corollary 3.5, and show that the conditions of Theorem 3.4 are of optimal form. Somewhat stronger conditions ensure the existence of a global solution, which is constructed as the limit of solutions of initial value problems at $r = r_0$, $r_0 \downarrow 0$ (Theorem 3.7). The arguments of this section are based on standard results from the theory of nonlinear parabolic equations, as described in [21] for example, together with a priori estimates for $\sup u$ and $\sup u^{-1}$, which are needed to control the parabolicity of (3.3). The behavior of the resulting solutions, asymptotically and at singular boundaries, is described in the next section.

As mentioned above, we consider (2.26) as a partial differential equation on $\mathbf{R}^+ \times S^2$ equipped with the product metric. Defining the auxiliary fields

$$\gamma = (1 - \tfrac{1}{2} \operatorname{div} \beta)^{-1}, \qquad \operatorname{div} \beta = \beta_{A|A} = \nabla^A \beta_A, \tag{3.1}$$

$$B = \tfrac{1}{2} |\operatorname{div} \beta|^2 + \tfrac{1}{2} |\beta_{(A|B)}|^2 - r\partial_r(\operatorname{div} \beta) + \beta_A (\operatorname{div} \beta)_{|A} - \tfrac{3}{2} \operatorname{div} \beta, \tag{3.2}$$

we rewrite (2.26) as

$$2r\partial_r u - 2\beta_A u_{|A} = \gamma u^2 \Delta u + (1 + \gamma B) u - \gamma (1 - \tfrac{1}{2} R_M r^2) u^3. \tag{3.3}$$

Now introducing $w = u^{-2}$ and $m = \frac{1}{2} r(1 - u^{-2})$, we have two useful equivalent forms:

$$(3.4)\qquad r\partial_r w - \beta_A w_{|A} = -\gamma u^{-1}\Delta u - (1+\gamma B)w + \gamma(1 - \tfrac{1}{2}R_M r^2),$$

$$(3.5)\ r\partial_r m - \beta_A m_{|A} = \tfrac{1}{2}r\gamma u^{-1}\Delta u - \gamma Bm + \tfrac{1}{2}r\gamma[B + (\gamma^{-1} - 1) + \tfrac{1}{2}R_M r^2].$$

The existence theorems are stated in terms of Hölder spaces, weighted to reflect the scaling properties of the parabolic equation. For any interval (open, half-open, or closed) $I \subset \mathbf{R}^+$, let $A_I = I \times S^2$. For any nonnegative integer $k$ and $0 < \alpha < 1$, define

$$\|f\|_{0,I} = \sup\{|f(r, \vartheta)|: (r, \vartheta) \in A_I\},$$

$$\langle f\rangle_{(\alpha),I} = \sup\left\{ r_2^{\alpha/2}\frac{|f(r_1, \vartheta_1) - f(r_2, \vartheta_2)|}{|r_1 - r_2|^{\alpha/2}} + \frac{|f(r_1, \vartheta_1) - f(r_2, \vartheta_2)|}{|\vartheta_1 - \vartheta_2|^{\alpha}} \right.$$
$$\text{for all } (r_i, \vartheta_i) \in A_I,\ i = 1, 2, \text{ such that}$$
$$\left. \tfrac{1}{2}r_1 < r_2 < 2r_1,\ r_1 \neq r_2,\ \vartheta_1 \neq \vartheta_2 \right\},$$

$$\|f\|_{(\alpha),I} = \|f\|_{0,I} + \langle f\rangle_{(\alpha),I}, \qquad \|f\|_{(k),I} = \sum_{|i|+2j\le k} \|\nabla^i (r\partial_r)^j f\|_{0,I},$$

$$\|f\|_{(k+\alpha),I} = \|f\|_{(k),I} + \sum_{|i|+2j=k} \langle \nabla^i (r\partial_r)^j f\rangle_{(\alpha),I}.$$

Here $\nabla$ and $|\vartheta_1 - \vartheta_2|$ denote the covariant derivative and geodesic distance, respectively, on $S^2$. For compact intervals $I \subset \mathbf{R}^+$, the parabolic Hölder space $C^{(k+\alpha)}(A_I)$ is the Banach space of continuous functions on $A_I$ with finite $\|\cdot\|_{(k+\alpha),I}$ norm, and for $I$ noncompact, $C^{(k+\alpha)}(A_I)$ is defined as the space of continuous functions which are norm-bounded on compact subsets of $I$. As usual, $C^{k,\alpha}(S^2)$ is the Hölder space on $S^2$ with norm $\|\cdot\|_{k,\alpha}$. Spaces of tensors satisfying Hölder conditions with respect to the standard metrics (and covariant derivatives) on $A_I$ and $S^2$ respectively, will be denoted similarly.

The normalizing factors in the definition of the Hölder norms $\|\cdot\|_{(k+\alpha),I}$ are chosen to provide simple behavior under dilation. For $\lambda > 0$ and $f \in C^{(k+\alpha)}(A_I)$, let $f_\lambda$ be the function

$$(3.6)\qquad f_\lambda(r, \vartheta) = f(\lambda r, \vartheta),$$

defined on $\lambda^{-1}I = \{r \in \mathbf{R}^+ : \lambda r \in I\}$. Then

$$\|f_\lambda\|_{(k+\alpha),\lambda^{-1}I} = \|f\|_{(k+\alpha),I}.$$

For any $f \in C^0(A_I)$ we define $f^*, f_*: I \to \mathbf{R}$ by

(3.7) $$f_*(r) = \inf\{f(r, \vartheta): \vartheta \in S^2\}, \qquad f^*(r) = \sup\{f(r, \vartheta): \vartheta \in S^2\}.$$

The local existence of solutions to (3.3) follows from the linear Schauder theory and a standard implicit function theorem argument.

**Proposition 3.1.** *Let $I = [r_0; r_1]$, $0 < r_0 < r_1 < \infty$, and let $\beta_A$ and $R_M$ be given in $A_I$ such that $\beta_A, \gamma, B, R_M \in C^{(\alpha)}(A_I)$, and*

(3.8) $$0 < \gamma_0 \le \gamma(r, \vartheta) \quad \forall (r, \vartheta) \in A_I,$$

*for some constant $\gamma_0 > 0$. Then for any initial condition*

(3.9) $$u(r_0, \vartheta) = \varphi(\vartheta), \qquad \vartheta \in S^2,$$

*where $\varphi \in C^{2,\alpha}(S^2)$ satisfies*

(3.10) $$0 < \delta_0 \le \varphi^{-2}(\vartheta) \le \delta_0^{-1}, \qquad \vartheta \in S^2,$$

*for some constant $\delta_0 > 0$, the initial value problem (3.3), (3.9) has a solution $u \in C^{(2+\alpha)}(A_{[r_0; r_0+T]})$ for some $T > 0$, where $T$ depends on $\gamma_0, \delta_0, r_0, \|\beta_A\|_{(\alpha),I}, \|\gamma\|_{(\alpha),I}, \|B\|_{(\alpha),I}, \|B_M\|_{(\alpha),I}$, and $\|\varphi\|_{2,\alpha}$.*

The basic uniform interior Hölder estimates which we need are summarized in

**Proposition 3.2.** *Let $I = [1; b]$ and $I' = [a; b]$ with $1 < a < b$, and suppose $u \in C^{(2+\alpha)}(A_I)$ is a solution of (3.3) in $A_I$, with source functions $\beta_A$ and $R_M$ such that $\beta_A, \gamma, B, R_M \in C^{(\alpha)}(A_I)$, $\nabla_A \gamma \in C^0(A_I)$, and*

(3.11) $$0 < \gamma_0 \le \gamma(r, \vartheta) \le \gamma_0^{-1} \quad \forall (r, \vartheta) \in A_I,$$

*for some constant $\gamma_0 > 0$. Further suppose there is a constant $\delta_0 > 0$ such that*

(3.12) $$0 < \delta_0 \le u^{-2}(r, \vartheta) \le \delta_0^{-1} \quad \forall (r, \vartheta) \in A_I.$$

*Then, with $m = \frac{1}{2} r(1 - u^{-2})$ as above, there is a constant $C$, depending on $a, b, \gamma_0, \delta_0, \|\beta_A\|_{(\alpha),I}, \|\gamma\|_{(\alpha),I}, \|\nabla_A \gamma\|_{0,I}, \|B\|_{(\alpha),I}$, and $\|R_M\|_{(\alpha),I}$, such that*

(3.13)
$$\|m\|_{(2+\alpha),I'} \le C(\|\beta_A\|_{(\alpha),I} + \|\gamma - 1\|_{(\alpha),I} + \|B\|_{(\alpha),I} + \|R_M\|_{(\alpha),I} + \|m\|_{0,I}).$$

*If $\beta_A, \gamma, B, R_M \in C^{(k+\alpha)}(A_I)$, $k \in \mathbf{Z}^+$, then there is a constant $C$, depending on $a, b, \delta_0, \gamma_0, \|\beta_A\|_{(k+\alpha),I}, \|\gamma\|_{(k+\alpha),I}, \|B\|_{(k+\alpha),I}$, and $\|R_M\|_{(k+\alpha),I}$, such that*

(3.14) $$\|m\|_{(k+2+\alpha),I'} \le C.$$

*Proof.* Let $I'' = [\frac{1}{2}(1+a), b]$, so $I' \subset I'' \subset I$. Writing (3.3) in divergence form, and using

$$\gamma u^2 \Delta u = \operatorname{div}(\gamma u^2 \nabla u) - u^2 \gamma_{|A} u_{|A} - 2\gamma u |\nabla u|^2,$$

we can apply [21, Theorem V.1.1] to obtain the Hölder estimate

$$\|u\|_{(\alpha'), I''} \leq C_1 \tag{3.15}$$

for some $0 < \alpha' < 1$, $\alpha' = \alpha'(\gamma_0, \delta_0)$, and constant $C_1$ depending on $a, b, \gamma_0, \delta_0$, $\|\beta_A\|_{0,I}$, $\|B\|_{0,I}$, and $\|R_M\|_{0,I}$. Without loss of generality we may assume $\alpha' \leq \alpha$. The usual Schauder interior estimates [21, Theorem IV.10.1] now give

$$\|u\|_{(2+\alpha'), I'} \leq C_2(C_1, \|\beta_A\|_{(\alpha), I}, \|\gamma\|_{(\alpha), I}, \|B\|_{(\alpha), I}, \|R_M\|_{(\alpha), I});$$

in particular, $u, u_{|A} \in C^{(\alpha)}(A_{I'})$ with uniform bounds. Noting the linear form of the lower order terms in (3.5) and that

$$r u^{-1} \Delta u = u^2 \Delta m + 3 u u_{|A} m_{|A},$$

from the Schauder estimates again we obtain (3.13), and (3.14) follows by the usual bootstrap argument. q.e.d.

The use of the variable $m$ rather than $u$, and the resulting linear form of the estimate (3.13), will be important in the proof of decay estimates. It is clear that in order to extend the interval of existence of the solution of Proposition 3.1, we need to control $u$ and $u^{-1}$. Suitable bounds will be derived from the next result.

**Proposition 3.3.** *Suppose* $u \in C^{(2+\alpha)}(A_{[r_0; r_1]})$, $0 < r_0 < r_1$, *is a positive solution of* (3.3). *Then for* $r_0 \leq r \leq r_1$ *we have*

(3.16)

$$r u^{-2}(r, \vartheta) \leq r_0 (u_*(r_0))^{-2} \exp\left(-\int_{r_0}^{r} (\gamma B)_*(t) \frac{dt}{t}\right) + \int_{r_0}^{r} \left(\gamma\left(1 - \frac{1}{2} R_M s^2\right)\right)^*(s) \exp\left(-\int_{s}^{r} (\gamma B)_*(t) \frac{dt}{t}\right) ds,$$

(3.17)

$$r u^{-2}(r, \vartheta) \geq r_0 (u^*(r_0))^{-2} \exp\left(-\int_{r_0}^{r} (\gamma B)^*(t) \frac{dt}{t}\right) + \int_{r_0}^{r} \left(\gamma\left(1 - \frac{1}{2} R_M s^2\right)\right)_*(s) \exp\left(-\int_{s}^{r} (\gamma B)^*(t) \frac{dt}{t}\right) ds.$$

*If we further assume* $\beta_A$ *and* $R_M$ *are defined on* $A_{\mathbf{R}^+}$ *such that the functions*

(3.18)
$$\delta^*(r) = \frac{1}{r}\int_0^r \left(\gamma\left(1-\frac{1}{2}R_M s^2\right)\right)^*(s)\exp\left(-\int_s^r (\gamma B)_*(t)\frac{dt}{t}\right) ds,$$

(3.19)
$$\delta_*(r) = \frac{1}{r}\int_0^r \left(\gamma\left(1-\frac{1}{2}R_M s^2\right)\right)_*(s)\exp\left(-\int_s^r (\gamma B)^*(t)\frac{dt}{t}\right) ds$$

*are defined and finite for all* $r \in \mathbf{R}^+$, *then the estimates* (3.16) *and* (3.17) *may be rewritten as*

(3.20)
$$u^{-2}(r,\vartheta) \le \delta^*(r) + \frac{r_0}{r}((u_*(r_0))^{-2} - \delta^*(r_0))\exp\left(-\int_{r_0}^r (\gamma B)_*(t)\frac{dt}{t}\right),$$

(3.21)
$$u^{-2}(r,\vartheta) \ge \delta_*(r) + \frac{r_0}{r}((u^*(r_0))^{-2} - \delta_*(r_0))\exp\left(-\int_{r_0}^r (\gamma B)^*(t)\frac{dt}{t}\right).$$

*Proof.* Applying the parabolic maximum principle to the equation (3.4) for $w = u^{-2}$ gives (at the maximum of $u(r,\vartheta)$)

$$rw_*'(r) \ge -(1+(\gamma B)^*)w_* + (\gamma(1-\tfrac{1}{2}R_M r^2))_*.$$

Setting $v(r) = r\exp(\int_{r_0}^r (\gamma B)^*(t)\,dt/t)w_*$, this can be rewritten as

$$v'(r) \ge \left(\gamma\left(1-\frac{1}{2}R_M r^2\right)\right)_*(r)\exp\left(\int_{r_0}^r (\gamma B)^*(t)\frac{dt}{t}\right),$$

and hence, since $v(r_0) = r_0 w_*(r_0)$,

$$w_*(r) \ge \frac{1}{r}\exp\left(-\int_{r_0}^r (\gamma B)^*(t)\frac{dt}{t}\right) \cdot\left(r_0 w_*(r_0) + \int_{r_0}^r \left(\gamma\left(1-\frac{1}{2}R_M s^2\right)\right)_*(s)\exp\left(\int_{r_0}^s (\gamma B)^*(t)\frac{dt}{t}\right) ds\right),$$

which is (3.17). Rearranging shows (3.17) is equivalent to (3.21) for $r_0 \le r \le r_1$, and (3.16) and (3.20) follow by similar arguments. q.e.d.

We now prove the existence of semiglobal solutions of the initial value problem and thereby the existence of a global solution (i.e., defined on all $A_{\mathbf{R}^+}$).

**Theorem 3.4.** *Suppose* $r_0 > 0$, *and* $\beta_A \in C^{(\alpha)}(A_{[r_0;\infty)})$, $R_M \in C^{(\alpha)}(A_{[r_0;\infty)})$ *are such that the functions* $\gamma$ *and* $B$, *which are defined from* $\beta_A$ *by* (3.1) *and* (3.2), *satisfy* $\gamma \in C^{(1+\alpha)}(A_{[r_0;\infty)})$, $B \in C^{(\alpha)}(A_{[r_0;\infty)})$ *and*

$$0 < \gamma_*(r) \le \gamma^*(r) < \infty \quad \forall r_0 \le r < \infty. \tag{3.22}$$

*Further assume the nonnegative constant* $K$, *defined by*

$$K = \sup_{r_0 \le r < \infty} \left\{ -\frac{1}{r_0} \int_{r_0}^{r} \left( \gamma \left( 1 - \frac{1}{2} R_M s^2 \right) \right)_* (s) \cdot \exp \left( \int_{r_0}^{s} (\gamma B)^*(t) \frac{dt}{t} \right) ds \right\}, \tag{3.23}$$

*satisfies*

$$K < \infty. \tag{3.24}$$

*Then for every* $\varphi \in C^{2,\alpha}(S^2)$ *such that*

$$0 < \varphi(\vartheta) < 1/\sqrt{K} \quad \forall \vartheta \in S^2, \tag{3.25}$$

*there is a unique positive solution* $u \in C^{(2+\alpha)}(A_{[r_0;\infty)})$ *of* (3.3) *with initial condition*

$$u(r_0, \cdot) = \varphi. \tag{3.26}$$

*Proof.* First observe that $u \in C^{(2+\alpha)}(A_{[r_0;\infty)})$ satisfies (3.3) if and only if $\tilde{u} \in C^{(2+\alpha)}(A_{[1;\infty)})$, $\tilde{u}(r, \vartheta) := u(r_0 r, \vartheta)$, satisfies[2]

$$2r\partial_r \tilde{u} - 2\tilde{\beta}_A \tilde{u}_{|A} = \tilde{\gamma}\tilde{u}^2 \Delta \tilde{u} + (1 + \tilde{\gamma}\tilde{B})\tilde{u} - \tilde{\gamma}(1 - \tfrac{1}{2}\tilde{R}_M r^2)\tilde{u}^3, \tag{3.27}$$

where

$$\tilde{\beta}_A(r, \vartheta) = \beta_A(r_0 r, \vartheta), \qquad \tilde{R}_M(r, \vartheta) = r_0^2 R_M(r_0 r, \vartheta),$$

and $\tilde{\gamma}$ and $\tilde{B}$, defined from $\tilde{\beta}_A$ by (3.1) and (3.2), also satisfy

$$\tilde{\gamma}(r, \vartheta) = \gamma(r_0 r, \vartheta), \qquad \tilde{B}(r, \vartheta) = B(r_0 r, \vartheta).$$

Denoting the estimating functions of Proposition 3.3 for (3.27) by $\tilde{\delta}^*(r)$ and $\tilde{\delta}_*(r)$, we see that

$$\tilde{\delta}^*(r) = \delta^*(r_0 r), \quad \tilde{\delta}_*(r) = \delta_*(r_0 r) \quad \forall 1 \le r < \infty,$$

[2]This scaling transformation is nothing more than a log $r$ translation of the partial differential equation on $\mathbf{R}^+ \times S^2$.

and it is similarly verified that

$$K = \sup_{1\le r<\infty} \left\{ -\int_1^r \left(\tilde{\gamma}\left(1-\frac{1}{2}\tilde{R}_M s^2\right)\right)_*(s) \exp\left(\int_1^s (\tilde{\gamma}\tilde{B})^*(t)\frac{dt}{t}\right) ds \right\}.$$

Hence the upper bound of (3.25) implies

$$0 < \tilde{\delta}_*(r) + \frac{1}{r}((\varphi^*)^{-2} - \tilde{\delta}_*(1)) \exp\left(-\int_1^r (\tilde{\gamma}\tilde{B})^*(t)\frac{dt}{t}\right) \tag{3.28}$$

for all $r \ge 1$. By Propositions 3.1 and 3.2, there is $T > 0$ and $\tilde{u} \in C^{(2+\alpha)}(A_{[1;1+T]})$ satisfying (3.27) with initial conditions

$$\tilde{u}(1, \cdot) = \varphi. \tag{3.29}$$

Furthermore, by Proposition 3.3 and (3.28) there are functions $0 < \delta_1(r) \le \delta_2(r) < \infty$, $1 \le r$, independent of $T$, such that

$$\delta_1(r) \le \tilde{u}(r, \vartheta) \le \delta_2(r) \quad \forall 1 \le r \le 1+T.$$

The precise forms of $\delta_1(r)$ and $\delta_2(r)$ follow from (3.20) and (3.21) and do not concern us. Let $U = \{t \in \mathbf{R}^+ : \exists \tilde{u} \in C^{(2+\alpha)}(A_{[1;1+t]})$ satisfying (3.3), (3.29)$\}$. The local existence Proposition 3.1 guarantees $U$ is open in $\mathbf{R}^+$ and from the interior estimate (3.13) of Proposition 3.2, we have an a priori estimate for $\|\tilde{u}(1+t, \cdot)\|_{2,\alpha}$ (observe that $I = [1; 1+t]$ is compact, hence there are $\gamma_0$ and $\delta_0$ satisfying (3.11) and (3.12) on $A_I$). By Proposition 3.1 the solution can be extended to $A_{[1,1+t+T]}$ for some constant $T$ independent of $\tilde{u}$, which shows that $U$ is closed. Hence $\tilde{u}$ extends to a semiglobal solution $\tilde{u} \in C^{(2+\alpha)}(A_{[1;\infty)})$ which is clearly unique, and the function $u(r, \vartheta) = \tilde{u}(r/r_0, \vartheta)$ is the required solution of (3.3), (3.26). q.e.d.

Note that if $R_M r^2 \le 2$, then $K = 0$ and the upper bound of (3.25) is trivally satisfied for all (positive) $\varphi \in C^{2,\alpha}(S^2)$. More generally, if $R_M$ has compact support, then $K < \infty$ and initial conditions $\varphi$ can be found, for which there is a semiglobal solution. This contrasts with the fact that if $R_M$ is sufficiently large and positive, then $\delta^*(r) < 0$ for some $r$ and therefore there can be *no* global solutions $u$ satisfying $u \to 1$ as $r \downarrow 0$. Geometrically, this says there are compactly supported functions $R_M \in C^\infty(\mathbf{R}^3)$ such that there is no complete QS metric having prescribed scalar curvature $R_M$. Likewise, blowup is also possible for the initial value problem, showing that the condition (3.25) is nearly optimal.

**Corollary 3.5.** *Let* $r_0$, $\beta_A$, $R_M$, *and* $K$ *be as given in Theorem* 3.4, *and suppose* $0 < K < \infty$. *If* $\varphi \in C^{2,\alpha}(S^2)$ *satisfies*

$$\varphi(\vartheta) > 1/\sqrt{K} \quad \forall \vartheta \in S^2, \tag{3.30}$$

*then there is* $T > 0$ *and* $u \in C^{(2+\alpha)}(A_{[r_0;r_0+T)})$ *satisfying the initial value problem* (3.3), (3.26), *such that*

$$\limsup_{r\to r_0+T}\{u^*(r)\} = \infty. \tag{3.31}$$

*Proof.* Arguing as before, the condition (3.30) ensures there is $r_1 > r_0$ such that

$$0 > r_0(\varphi_*)^{-2}\exp\left(-\int_{r_0}^{r_1}(\gamma B)_*(t)\,\frac{dt}{t}\right) + \int_{r_0}^{r_1}\left(\gamma\left(1-\frac{1}{2}R_M s^2\right)\right)^*(s)\exp\left(-\int_s^{r_1}(\gamma B)_*(t)\,\frac{dt}{t}\right)ds.$$

The lower bound (3.16) shows there can be no solution of (3.3), (3.26) on $A_{[r_0;r_1]}$, hence there is a maximal $T \in (0; r_1 - r_0)$ and a solution $u \in C^{(2+\alpha)}(A_{[r_0;r_0+T)})$. Maximality and Proposition 3.2 show that the lower bound (3.12) must fail, showing (3.31). q.e.d.

Another immediate corollary of Theorem 3.4 is the existence of extension metrics having boundary $S_{r_0}$ with prescribed (positive) mean curvature.

**Corollary 3.6.** *Let* $r_0$, $\beta_A$, $R_M$, *and* $K$ *be as given in Theorem* 3.4. *Suppose* $h \in C^{2,\alpha}(S^2)$ *satisfies*

$$\gamma(r_0, \vartheta)h(\vartheta) > 2\sqrt{K}/r_0 \quad \forall\vartheta \in S^2. \tag{3.32}$$

*Then there is a QS metric with scalar curvature* $R_M$ *and shear vector* $\beta_A$, *having boundary* $S_{r_0} \cong r_0^2 S^2$ *with mean curvature* $h$.

*Proof.* Let $\varphi(\vartheta) = 2/(r_0\gamma(r_0, \vartheta)h(\vartheta))$; then (3.32) is equivalent to $\varphi^* < 1/\sqrt{K}$. Theorem 3.4 now constructs a solution $u \in C^{(2+\alpha)}(A_{[r_0;\infty)})$ to the initial value problem (3.3), (3.26), and the resulting QS metric has boundary $S_{r_0}$ with mean curvature $h(\vartheta)$ by (2.17). q.e.d.

The semiglobal existence Theorem 3.4 can be used to construct global solutions, without specified initial conditions.

**Theorem 3.7.** *Let* $\beta_A$ *and* $R_M$ *be given on* $A_{\mathbf{R}^+} = \mathbf{R}^+ \times S^2$ *such that* $\beta_A \in C^{(\alpha)}(A_{\mathbf{R}^+})$, $\gamma \in C^{(1+\alpha)}(A_{\mathbf{R}^+})$, $B \in C^{(\alpha)}(A_{\mathbf{R}^+})$, *and* $R_M \in C^{(\alpha)}(A_{\mathbf{R}^+})$, *where* $\gamma$ *and* $B$ *are the derived functions defined by* (3.1) *and* (3.2). *Suppose* $\gamma$, $B$, *and* $R_M$ *satisfy the global bounds*

$$0 < \gamma_*(r) \le \gamma^*(r) < \infty, \tag{3.33}$$

$$0 < \delta_*(r) \le \delta^*(r) < \infty \tag{3.34}$$

*for all* $r > 0$, *where* $\delta_*(r)$ *and* $\delta^*(r)$ *are defined by* (3.18) *and* (3.19). *Then there is a solution* $u \in C^{(2+\alpha)}(A_{\mathbf{R}^+})$ *of* (3.3), *such that for all* $(r, \vartheta) \in \mathbf{R}^+ \times S^2$

$$1/\sqrt{\delta^*(r)} \le u(r, \vartheta) \le 1/\sqrt{\delta_*(r)}. \tag{3.35}$$

*Proof.* Let $\varphi_\varepsilon \in C^{2,\alpha}(S^2)$, $0 < \varepsilon < 1$, be any family of functions satisfying

$$\delta_*(\varepsilon) \le \varphi_\varepsilon^{-2}(\vartheta) \le \delta^*(\varepsilon), \tag{3.36}$$

and let $u^{(\varepsilon)}$ be the solution of (3.3) on $A_{[\varepsilon, \infty)}$ with initial condition $\varphi_\varepsilon$ (the existence of $u^{(\varepsilon)}$ follows from Theorem 3.4). From Proposition 3.3 we have

$$\delta_*(r) \le (u^{(\varepsilon)}(r, \vartheta))^{-2} \le \delta^*(r), \qquad \varepsilon \le r < \infty, \tag{3.37}$$

for all $0 < \varepsilon < 1$. Now suppose $r_0 > 0$ and $u \in C^{(2+\alpha)}(A_I)$, $I = [r_0; 4r_0]$, is a solution of (3.3) satisfying

$$\delta_*(r) \le u^{-2}(r, \vartheta) \le \delta^*(r) \quad \forall r \in I, \tag{3.38}$$

and define $\tilde{u}(r, \vartheta) = u(r/r_0, \vartheta)$. By applying Proposition 3.2 to $\tilde{u}$ on the interval [1;4] and then rescaling back, from (3.13) we obtain an estimate of the form

$$\|u\|_{(2+\alpha), I'} \le C, \qquad I' = [2r_0; 4r_0], \tag{3.39}$$

where $C$ is a constant which does not depend on $u$. (For later application, observe that (3.13) rescales to give more precisely

(3.40)

$$\|m/r\|_{(2+\alpha), I'} \le C \Big\{ \|\beta_A\|_{(\alpha), I} + \|B\|_{(\alpha), I} + \|\gamma - 1\|_{(\alpha), I} + \|R_M r^2\|_{(\alpha), I} + \sup_{r \in I}(1 - \delta_*(r)) + \inf_{r \in I}(1 - \delta^*(r)) \Big\},$$

where $C$ depends on $\sup_I\{\gamma^*(r), \gamma_*^{-1}(r)\}$, $\sup_I\{\delta^*(r), \delta_*^{-1}(r)\}$, $\|\beta_A\|_{(\alpha), I}$, $\|\gamma\|_{(1+\alpha), I}$, $\|B\|_{(\alpha), I}$, and $\|R_M r^2\|_{(\alpha), I}$.)

Applying (3.39) to $u^{(\varepsilon)}$ shows, by Ascoli-Arzela, there is a sequence $\varepsilon_j \downarrow 0$ such that the sequence $\{u^{(\varepsilon_j)}\}$ converges uniformly in $C^{(2+\alpha)}(A_I)$ for any compact interval $I \subset \mathbf{R}^+$ to the required solution $u \in C^{(2+\alpha)}(\mathbf{R}^+ \times S^2)$. q.e.d.

Note that the global solution constructed here need not be unique, due to the freedom in choosing the initial data $\varphi_\varepsilon \in C^{2,\alpha}(S^2)$, $\varepsilon > 0$, for the approximating solutions $u^{(\varepsilon)}$. Uniqueness in general requires greater control on the behavior of $\gamma B$ and $u$ as $r \downarrow 0$ (see, for example, Theorem 4.3).

## 4. Asymptotic metric behavior

The asymptotic behavior (as $r \to 0, \infty$) of the global solution constructed above is not controlled, since the source functions $\beta_A$ and $R_M$ are restricted only by the (mild) conditions (3.33) and (3.34). There are four geometrically natural asymptotic boundary conditions:

(i) regular at the center ($u = 1 + O(r^2)$ as $r \downarrow 0$),

(ii) minimal surface ("black hole") interior boundary ($u^{-1} \to 0$ as $r \downarrow r_0 > 0$),

(iii) asymptotically Euclidean ($u = 1 + O(r^{-1})$ as $r \to \infty$),

(iv) asymptotically hyperbolic ($u = r^{-1} + O(r^{-2})$ as $r \to \infty$).

In this section we describe conditions on $\beta_A$ and $R_M$ which ensure existence of solutions satisfying the boundary conditions (i), (ii), (iii). The basic tool is a dilation-invariance property of (3.3) which allows us to deduce decay estimates from local estimates. This is sufficient to prove the curvature is bounded across $r = 0$ (Theorem 4.3), and asymptotic flatness (Theorem 4.2), under suitable decay conditions on $\beta_A$ and $R_M$. These asymptotic conditions could be weakened to allow $u = 1 + O(r^{-\alpha})$, $\alpha > 0$—this is left as an exercise for the interested reader. A restatement of these results, in rectangular rather than spherical polar coordinates, is given by Theorem 4.5.

The black hole boundary result (Theorem 4.6) uses a curious desingularization (4.47), which transforms the equation into a similar equation with boundary conditions posed at $r = 0$. The discussion following Theorem 4.5 shows that compact minimal surfaces form a natural obstruction to the existence of QS coordinates in general metrics. Although global existence in the asymptotically hyperbolic case (iv) follows from Theorem 3.7 and it is easy to see $ru \to 1$ as $r \to \infty$, the estimate (3.40) needed to control curvature is of little value, since the ellipticity of (3.3) degenerates with $u$. This case will be considered elsewhere.

**Lemma 4.1.** *Suppose* $y \in L^1([1;\infty))$. *Then there is a constant* $C$ *depending on* $y$ *such that*

$$(4.1)\qquad 1-\frac{C}{r}\le\frac{1}{r}\int_1^r\exp\left(-\int_s^r y(t)\,\frac{dt}{t}\right)\,ds\le 1+\frac{C}{r}$$

*for all* $r\ge 1$.

*Proof.* Since $\int_1^\infty |y|\,dt<\infty$, there is $r_0>1$ such that $\int_{r_0}^\infty |y|\,dt\le 1$. Using

$$|e^\eta-1|\le 2|\eta|\quad\text{for } |\eta|\le 1,$$

we see that

$$\begin{aligned}
&\left|\int_1^r\exp\left(-\int_s^r y(t)\,\frac{dt}{t}\right)\,ds-r\right|\\
&\quad\le\left|\int_1^{r_0}\exp\left(-\int_s^r y(t)\,\frac{dt}{t}\right)\,ds+r_0\right.\\
&\qquad\left.+\int_{r_0}^r\left(\exp\left(-\int_s^r y(t)\,\frac{dt}{t}\right)-1\right)\,ds\right|\\
&\quad\le C+2\int_{r_0}^r\int_s^r|y(t)|\,\frac{dt}{t}\,ds\\
&\quad\le C+2\int_{r_0}^r|y(t)|\left(1-\frac{r_0}{t}\right)\,dt\le C,
\end{aligned}$$

which implies (4.1). q.e.d.

Associated with the quasi-spherical coordinates $(r,\vartheta)$ on $\mathbf{R}^+\times S^2$ are *natural rectangular coordinates* $(x^i)=(x,y,z)$, using the usual spherical polar/rectangular coordinate transformations. (The transformation is determined only up to a rigid rotation, but this does not matter.) The rectangular coordinates define an embedding $(x^i)\colon\mathbf{R}^+\times S^2\to\mathbf{R}^3$, and we compare $g$ with the pullback of the flat metric $|dx|^2$ of $\mathbf{R}^3$ under this map. Since $|dx|^2=dr^2+r^2\sum\sigma_A^2$, we have

$$\begin{aligned}
g-|dx|^2&=(g_{ij}-\delta_{ij})\,dx^i\,dx^j\\
&=(u^2+\beta^2-1)\,dr^2+2r\beta_A\sigma_A\,dr,
\end{aligned}$$

where $\beta^2=\sum\beta_A^2$ is the length squared in the $S^2$ metric of the 1-form $\beta_A\sigma_A$. Defining $\beta_i$, $i=1,2,3$, by the relations

$$(4.2)\qquad \beta_i x^i=0,\qquad \beta_A\sigma_A=\beta_i\,d\theta_i,$$

where $\theta_i=x^i/r$, $d\theta_i=r^{-1}\theta_{ij}\,dx^j=r^{-1}(\delta_{ij}-\theta_i\theta_j)\,dx^j$, and $\beta^2=\sum\beta_A^2=\sum\beta_i^2$ since $d\sigma^2=\sum\sigma_A^2=\sum d\theta_i^2$, we see that

$$(4.3)\qquad g_{ij}=\delta_{ij}+(u^2+\beta^2-1)\theta_i\theta_j+\beta_i\theta_j+\beta_j\theta_i.$$

Let us adopt as a definition of asymptotic flatness the conditions

$$|g_{ij} - \delta_{ij}| + r|\partial_i g_{jk}| \le C/r, \qquad i, j, k = 1, 2, 3, \tag{4.4}$$

for $r > r_0$ and constants $r_0$ and $C$. Although this is not the weakest possible definition [3], by assuming further the condition $R_M \in L^1(M)$, we can ensure that the ADM mass is well defined. From (4.3) and (4.2) it is clear that (4.4) will be satisfied if there are constants $C$ and $r_0$ such that for all $r > r_0$,

$$|\beta_A| + |r\partial_r \beta_A| + |\nabla_A \beta_B| \le C/r, \tag{4.5}$$

$$|u^2 - 1| + |r\partial_r u| + |\nabla_A u| \le C/r. \tag{4.6}$$

The conditions on $\beta_A$ can be imposed a priori, and the estimates for $u$ are shown in the following result.

**Theorem 4.2.** *Let* $u \in C^{(2+\alpha)}(A_{[r_1, \infty)})$ *be a solution of* (3.3) *satisfying*

$$1/\sqrt{\delta^*(r_1)} \le u(r_1, \vartheta) \le 1/\sqrt{\delta_*(r_1)} \tag{4.7}$$

*and suppose there is a constant* $C > 0$ *such that for all* $r \ge 2r_1$ *and* $I_r = [\frac{1}{2}r; 2r]$,

$$|\beta_A|_{(2+\alpha), I_r} + |r\partial_r \beta_A|_{(1+\alpha), I_r} + |R_M r^2|_{(\alpha), I_r} \le C/r. \tag{4.8}$$

*Further assume*

$$\int_{r_0}^{\infty} (|r\partial_r \operatorname{div} \beta|^*(r) + |\operatorname{div} \beta|^*(r))\, dr < \infty, \tag{4.9}$$

$$\int_{r_0}^{\infty} |R_M|^*(r) r^2\, dr < \infty. \tag{4.10}$$

*Then the asymptotic flatness condition* (4.4) *is satisfied, and the curvature tensor is Hölder continuous and decays as*

$$|\operatorname{Riem}| \le C/r^3, \tag{4.11}$$

*where* $|\operatorname{Riem}| = (g^{ii'} g^{jj'} g^{kk'} g^{ll'} R_{ijkl} R_{i'j'k'l'})^{1/2}$. *The total* (*ADM*) *mass* ([1], [3]) *of* $(M, g)$ *is well defined and given by*

$$m_{\text{ADM}} = \frac{1}{4\pi} \lim_{r\to\infty} \oint_{S^2} m(r, \vartheta)\, d\sigma. \tag{4.12}$$

*Proof.* From (4.8) and (4.9)

$$\int_{r_1}^{\infty} |B|^*(r)\, dr < \infty, \tag{4.13}$$

so (4.10) and Lemma 4.1 give

$$1 - C/r \le \delta_*(r) \le \delta^*(r) \le 1 + C/r \tag{4.14}$$

for all $r \ge r_1$. Now $m$ scales as $m_\lambda(r) = m(\lambda r)/\lambda$, so the scaling argument of Theorem 3.7 and the bound (3.13) yield

$$\begin{aligned} \|m\|_{(2+\alpha), I'_r} \le C\|m\|_{0, I_r} + Cr\{\|\beta_A\|_{(\alpha), I_r} + \|B\|_{(\alpha), I_r} \\ + \|\gamma - 1\|_{(\alpha), I_r} + \|R_M r^2\|_{(\alpha), I_r}\}, \end{aligned} \tag{4.15}$$

where $I'_r = [r; 2r]$, and $C$ is bounded independently of $u$ and $r$ by (4.14) and the decay assumptions (4.9). Now (4.14) controls $\|m\|_{0, I_r}$ and the decay (4.8) controls the second term of (4.15), giving the uniform bound

$$\|m\|_{(2+\alpha), I'_r} \le C \tag{4.16}$$

for all $r \ge r_1$. Expressing this in terms of $u$ and derivatives implies

$$\|1 - u^{-2}\|_{(\alpha), I'_r} + \|r\partial_r u\|_{(\alpha), I'_r} + \|\nabla u\|_{(\alpha), I'_r} + \|\nabla^2 u\|_{(\alpha), I'_r} \le C/r. \tag{4.17}$$

The estimates for $u^{-2}$ and $\nabla u$ show $g$ is asymptotically Euclidean in the sense of definition (4.4), and the estimate for $\nabla^2 u$, together with the expressions (2.23)–(2.25) for Ric, shows that $\mathrm{Ric} \in C^{0,\alpha}(A_{[r_1;\infty)})$ and $|\mathrm{Ric}(r, \vartheta)| \le C/r^3$. Since $M$ is 3-dimensional, this controls the full curvature tensor.

It follows from [3, §4] that the ADM mass

$$m_{\mathrm{ADM}} = \frac{1}{16\pi} \oint_{S_\infty} (\partial_i g_{ij} - \partial_j g_{ii})\, dS^j \tag{4.18}$$

is uniquely defined, since $\partial g \in L^2(\mathbf{R}^3 \backslash \mathbf{B}(0, r_1))$ and $R_M \in L^1(\mathbf{R}^3 \backslash \mathbf{B}(0, r_1))$. Regarding the sphere at infinity $S_\infty$ as the limit of the foliation spheres $S_r$ and noting that

$$dS_j = \theta_j\, dS = r^2 \theta_j\, d\sigma,$$

where $d\sigma$ is the volume form on $S^2$, we find that

$$(\partial_i g_{ij} - \partial_j g_{ii})\theta_j = \frac{2}{r}(u^2 + \beta^2 - 1) + \partial_i \beta_i.$$

A short calculation using the relations (4.2) shows

$$r\partial_i \beta_i = \operatorname{div} \beta = \beta_{A|A}, \tag{4.19}$$

whereupon $\oint \partial_i \beta_i = 0$. Since $\beta^2 \le C/r^2$, the mass integral reduces to

$$\frac{1}{16\pi} \lim_{r\to\infty} \oint_{S_r} \frac{2}{r}(u^2 - 1) r^2 \, d\sigma$$

and (4.12) follows by noting from (4.16) that $u^2 = 1 + 2m/r + O(r^{-2})$. The limit in (4.12) can be shown to exist directly from (3.5), which gives

$$\frac{d}{dr} \oint m = \frac{1}{2} \oint \gamma u^{-1} \Delta u - \frac{1}{r} \oint (\operatorname{div} \beta + \gamma B) m + \frac{1}{4} \oint (2\gamma B + \operatorname{div} \beta + \gamma R_M r^2). \tag{4.20}$$

Here and henceforth we use $\oint$ to denote the integral over $S^2$ with the standard measure $d\sigma$. Since $u^{-1}\nabla u = u^2 \nabla m/r$, $m$ and $\nabla m$ are bounded, and $\operatorname{div} \beta$, $B = O(r^{-1})$, the first two terms are integrable on $(r_1; \infty)$, while the integrability of the final term is ensured by the slightly stronger, but evidently necessary, decay conditions (4.9) and (4.10). q.e.d.

By means of very similar arguments we establish regularity across the center $r = 0$, using the rectangular coodinates $(x^i)$: $\mathbf{R}^+ \times S^2 \to \mathbf{R}^3$ to define the differentiable structure on $\mathbf{R}^+ \times S^2 \cup \{\text{centre}\}$.

**Theorem 4.3.** *Let $r_0 > 0$, assume $\beta_A$ and $R_M$ satisfy*

$$\|r^{-2} \beta_A\|_{(3+\alpha), (0; r_0]} + \|R_M\|_{(\alpha), (0; r_0]} \le C, \tag{4.21}$$

*and let $u \in C^{(2+\alpha)}(A_{(0; r_0)})$ be a solution constructed by Theorem 3.7. Then $u$ is unique in the class of solutions satisfying*

$$|u - 1| \le C r^{\varepsilon}$$

*for some $\varepsilon > 0$, $C$ and all $0 < r < r_0$, and the coefficients $g_{ij}$ of the resulting quasi-spherical metric $g$ in natural rectangular coordinates $(x^i)$ satisfy $g_{ij} \in C^1(\mathbf{B}(0, r_0)) \cap C^{2,\alpha}(\mathbf{B}(0, r_0) \setminus \{0\})$, where $\mathbf{B}(0, r_0) \subset \mathbf{R}^3$ is the ball of radius $r_0$ and center $0$, and we define $g_{ij}(0) = \delta_{ij}$. Furthermore, there is a constant $C$ such that for all $0 < r \le r_0$,*

$$|g_{ij} - \delta_{ij}| + r|\partial_i g_{jk}| \le C r^2, \tag{4.22}$$

$$|\operatorname{Ric}| \le C. \tag{4.23}$$

*Proof.* The decay bounds (4.21) imply there is $0 < r_1 \le r_0$ such that

$$|B(r, \vartheta)| \le C r^2, \qquad |\gamma(r, \vartheta) - 1| \le \tfrac{1}{3} \tag{4.24}$$

for $0 < r \le r_1 \le r_0$, and from Lemma 4.1 we have

$$|\delta^*(r) - 1| + |\delta_*(r) - 1| \le C r^2. \tag{4.25}$$

The method of Theorem 3.7 constructs a solution $u$ bounded by $\delta^*$ and $\delta_*$, and (3.40) gives

$$|m|_{(2+\alpha),[r;2r]} \leq Cr^3 \tag{4.26}$$

for $r \leq \frac{1}{2}r_1$, which implies the required bounds on $g_{ij}$ and $\partial_i g_{jk}$. The curvature bound follows from (2.23), as before.

To show uniqueness, suppose $u_1$ and $u_2$ are two solutions of (3.3) and set $v = m_1 - m_2$, where $m_i = \frac{1}{2}r(1 - u_i^{-2})$, $i = 1, 2$. From (3.5)

$$r\partial_r v - \beta_A \nabla_A v = \tfrac{1}{2} r\gamma(u_1^{-1}\Delta u_1 - u_2^{-1}\Delta u_2) - \gamma B v, \tag{4.27}$$

and multiplying by $v$ and integrating over $S^2$ gives (setting $\oint = \int_{S^2}$)

$$\begin{aligned} r\frac{d}{dr}\oint v^2 = &- \oint(\operatorname{div}\beta + \gamma B)v^2 + r\oint \gamma\frac{u_1^2 u_2^2}{r^2} \\ &\cdot\Big\{(r - m_1 - m_2)(-|\nabla v|^2 - v\gamma^{-1}\nabla_A\gamma\nabla_A v + r^{-2}u_1^2u_2^2|\nabla v|^2 v^2) \\ &\quad + \nabla_A(m_1 + m_2)\Big(u_1^{-1}\nabla_A u_1 + u_2^{-1}\nabla_A u_2 \\ &\qquad + r^{-2}u_1^2u_2^2 v\nabla_A v - \gamma^{-1}\nabla_A\gamma\Big)v^2\Big\}. \end{aligned}$$

The various terms are estimated using

- an easy consequence of (4.21) to control $\operatorname{div}\beta$, $\gamma$, $\nabla\gamma$, $\gamma B$:

$$|\operatorname{div}\beta| + |\nabla\gamma| + |\gamma B| \leq Cr;$$

- the asymptotic assumption $u_1, u_2 \to 1$ as $r \to 0$ to control $u_1^2, u_2^2$ terms;
- the Schwarz inequality

$$v\gamma^{-1}\nabla_A\gamma\nabla_A v \leq |\nabla v|^2 + \tfrac{1}{4}\gamma^{-2}|\nabla\gamma|^2 v^2;$$

- the identity $u^{-1}\nabla u = u^2\nabla m/r$ and $|\nabla v| \leq |\nabla m_1| + |\nabla m_2|$ to control terms in $\nabla u$ and $\nabla v$, giving finally

$$r\frac{d}{dr}\oint v^2 \leq C\oint\{r + (|\nabla m_1|^2 + |\nabla m_2|^2)(r^{-2} + |v|r^{-3})\}v^2. \tag{4.28}$$

Now if $u_1$ and $u_2$ satisfy

$$|u_1 - 1| + |u_2 - 1| \leq Cr^{\varepsilon}$$

for some $\varepsilon > 0$ and all $0 < r \leq r_0$, then

$$|m_1| + |m_2| \leq Cr^{1+\varepsilon}$$

and the rescaling estimate (3.40) gives

$$|\nabla m_1| + |\nabla m_2| \le Cr^{1+\varepsilon}.$$

Using $|v| \le |m_1| + |m_2| \le Cr^{1+\varepsilon}$ we find therefore

$$\frac{d}{dr}\oint v^2 \le Cr^{-1+2\varepsilon}\oint v^2.$$

Solving this differential inequality yields

$$\oint_{S_r} v^2 \le \exp(C\varepsilon^{-1}(r^{2\varepsilon} - t^{2\varepsilon}))\oint_{S_t} v^2$$

for all $0 < t \le r \le r_0$. Since $v \to 0$ as $r \downarrow 0$, the right-hand side goes to 0 as $t \downarrow 0$, hence $v \equiv 0$, showing uniqueness. q.e.d.

The regularity of $g_{ij}$ can be improved slightly by using the standard harmonic coordinate argument and the boundedness of the Ricci tensor (cf. [3, Proposition 3.3]), but it does not seem possible to assert higher regularity (in particular, continuous curvature) using only the rather soft scaling method given here. However, one situation where regularity at the centre is elementary occurs when $\beta_A$ and $R_M$ vanish identically for small $r$:

**Corollary 4.4.** *Suppose $\beta_A$ and $R_M$ satisfy the global existence conditions of Theorem 3.7 on $\mathbf{R}^+ \times S^2$, the decay conditions (4.21) on $A_{(0;r_0]}$, and*

$$\operatorname{spt}(\beta_A) \cup \operatorname{spt}(R_M) \subset A_{[r_1;\infty)} \tag{4.29}$$

*for some $0 < r_1 \le r_0$. Then the metric $(\mathbf{R}^3, g)$ constructed by Theorem 3.7 is metrically flat on the region $\mathbf{B}(0, r_1)$.*

*Proof.* Clearly $\delta^*(r) = \delta^*(r) = 1$ for $0 < r \le r_1$, and hence the global solution satisfies $m(r, \vartheta) = 0$ for $0 < r \le r_1$.

**Remark.** By specializing to $R_M \equiv 0$ and requiring that $\beta_A$ satisfy the decay conditions (4.9) and (4.10), this corollary provides examples of asymptotically flat, time-symmetric initial data sets for the vacuum Einstein equations on $\mathbf{R}^3$, containing regions of vanishing spacetime curvature.

It may be helpful at this point to collect and restate a version of these results for data given in terms of rectangular coordinates on $\mathbf{R}^3$.

**Theorem 4.5.** *Suppose $\beta_i, R_M \in C^\infty(\mathbf{R}^3)$ are given such that $\sum \beta_i(x)x^i = 0$ for all $x \in \mathbf{R}^3$, and*

$$r\sum \partial_i\beta_i(x) < 2, \tag{4.30}$$

$$R_M(x)r^2 < 2, \tag{4.31}$$

*where* $r = |x|$ *and* $\partial_i = \partial/\partial x^i$. *Further assume* $\beta_i$ *and* $R_M$ *satisfy the following asymptotic conditions* (*for some constants* $0 < r_0 \le r_1 < \infty$):

(i) *for* $r_1 \le r < \infty$,

$$|R_M| + r|\partial R_M| \le C_1/r^4, \tag{4.32}$$

$$|\beta| + r|\partial\beta| + r^2|\partial^2\beta| + r^3|\partial^3\beta| \le C_2/r, \tag{4.33}$$

$$|\partial_i\beta_i| + r|\partial_r(\partial_i\beta_i)| \le C_3/r^3; \tag{4.34}$$

(ii) *for* $0 < r \le r_0$,

$$|\beta| + r|\partial\beta| \le C_4 r^2, \tag{4.35}$$

*where* $\partial_r = r^{-1}x^i\partial_i$ *and*

$$|\partial^k\beta|^2 = \sum_{i_1, \cdots, i_k, j} |\partial_{i_1}\cdots\partial_{i_k}\beta_j(x)|^2.$$

*Then there is* $u \in C^\infty(\mathbf{R}^3\backslash\{0\})$ *satisfying* (3.3) *and*

$$u(x) = 1 + O(r^2) \quad \text{as } r \downarrow 0,$$

*such that the resulting quasi-spherical metric* (4.3) *satisfies* $g_{ij} \in C^\infty(\mathbf{R}^3\backslash\{0\}) \cap C^{1,\alpha}(\mathbf{R}^3)$. *Furthermore, the curvature tensor of* $g$ *is defined and bounded almost everywhere,*

$$|\mathrm{Ric}(x)| \le C \quad \forall x \in \mathbf{R}^3\backslash\{0\},$$

*and* $g$ *has asymptotic decay*

$$|g_{ij} - \delta_{ij}| + r|\partial_i g_{jk}| \le C/r, \tag{4.36}$$

$$|\mathrm{Ric}| \le C/r^3, \tag{4.37}$$

*with finite ADM mass* (4.12).

**Remark.** Condition (4.31) is stronger than necessary, and could be replaced by the weaker but less intuitive condition (3.34).

*Proof.* Formula (4.2) serves to cast $\beta_i$ into angular form, and from (4.19) it follows that $\operatorname{div}\beta = r\partial_i\beta_i$, so (4.30) ensures $\gamma < \infty$, and thus (3.33) holds. To show (3.34) calculate

$$|\beta_{(A|B)}|^2 = r^2(|\partial_{(i}\beta_{j)}|^2 - \tfrac{1}{2}|\partial_r\beta_i|^2) + \tfrac{1}{2}r\partial_r\beta^2, \tag{4.38}$$

whence $|B| \le Cr^2$, while (4.31) gives $\delta_*(r) > 0$ and (3.34) follows. Theorem 3.7 then yields a global solution $u(x)$, $x \ne 0$, of (3.3) satisfying

$$0 < \delta_*(r) \le u^{-2}(x) \le \delta^*(r) < \infty.$$

The asymptotic conditions now ensure that Theorems 4.2 and 4.3 apply to give regularity about $r = 0$, and asymptotic flatness. q.e.d.

Since the direction of increasing $r$ is the direction of propagation of the parabolic equation (3.3), it is clear the assumption $dr \neq 0$ is an essential ingredient of the quasi-spherical construction. The prototypical example of the breakdown of this assumption is the Schwarzschild metric

$$ds^2_{\text{Schw}} = -\frac{dr^2}{1 - 2M/r} + r^2\, d\sigma^2$$

on $S^2 \times [2M\,;\infty)$, where $M$ is a positive constant. The manifold obtained by doubling across the totally geodesic 2-sphere $r = 2M$ is smooth, complete, and scalar-flat, and $r$ is globally defined and smooth. However, $r$ fails to be a coordinate across $r = 2M$, and the breakdown $dr \to 0$ corresponds to $u \to \infty$. This suggests the boundary condition

$$u^{-1}(r_0\,, \vartheta) = 0 \tag{4.39}$$

(or equivalently, $m(r_0\,, \vartheta) = \frac{1}{2}r_0)$, which by (2.17) implies that the boundary $S_{r_0}$ is a minimal 2-sphere. For time-symmetric initial data sets in general relativity, the minimal surface boundary condition corresponds to an apparent horizon ("black hole," roughly speaking) at $r = r_0$, and the following result constructs such solutions.

Note that, provided the QS geometry is bounded (in the sense that the shear vector $\beta_A$ is smooth and bounded across $r = r_0$), this boundary condition also implies that the second fundamental form of the boundary sphere vanishes identically. Although it may be possible to produce QS solutions with nontotally geodesic minimal surface boundary, by allowing certain components of $\beta_A$ to blow up appropriately, it appears that in such cases the Ricci curvature will not be bounded at the horizon.

The restriction $dr \neq 0$ indicates the quasi-spherical technique is naturally limited to constructing metrics in the "exterior" of all minimal surfaces. For example, if $K \subset M$ is a compact subset of a 3-manifold $M$ such that $M \backslash K$ admits a QS foliation (with $\beta_A$ smooth and $\operatorname{div}\beta < 2$), then by the maximum principle, any closed compact minimal surface in $M$ necessarily lies inside $K$.

**Theorem 4.6.** *Let $r_0 > 0$, and let $\beta_A$ and $R_M$ be given such that*

$$R_M \in C^{(\alpha)}(A_{[r_0\,;\infty)}), \tag{4.40}$$

$$\beta_A \in C^{(3+\alpha)}(A_{[r_0\,;\infty)}). \tag{4.41}$$

*Further suppose that* $R_M$ *and* $\operatorname{div}\beta$ *satisfy, for* $r_0 \le r < \infty$,

$$R_M r^2 < 2, \tag{4.42}$$

$$\operatorname{div}\beta < 2. \tag{4.43}$$

*Then there is* $u^{-1} \in C^{(2+\alpha)}(A_{(r_0;\infty)})$ *such that the quasi-spherical metric* $g$ *on* $A_{[r_0;\infty)}$ *has curvature uniformly bounded on* $A_{[r_0;2r_0]}$ *with totally geodesic boundary* $S_{r_0}$,

$$\mathrm{II}_{S_{r_0}} = 0.$$

*Let* $0 < \eta < 1$ *be such that*

$$1 - \eta < [\gamma(1 - \tfrac{1}{2}R_M r^2)]_{r=r_0} < (1-\eta)^{-1}. \tag{4.44}$$

*Then there is* $r_0' > r_0$ *such that for* $r_0 \le r \le r_0'$,

$$\frac{r - r_0}{r}(1-\eta) \le u^{-2}(r) \le \frac{r-r_0}{r}(1-\eta)^{-1} \tag{4.45}$$

$$\Leftrightarrow r_0 - \frac{\eta}{1-\eta}(r - r_0) \le 2m \le r_0 + \eta(r - r_0). \tag{4.46}$$

*Proof.* For clarity in the following computations we suppress explicit mention of the $\vartheta$-dependence. Defining, for $r > 0$,

$$\tilde{u}(r) = \sqrt{\frac{r}{r + r_0}}\, u(r + r_0), \tag{4.47}$$

the evolution equation can be rewritten as

$$2r\partial_r \tilde{u} - 2\tilde{\beta}_A \tilde{u}_{|A} = \tilde{\gamma}\tilde{u}^2 \Delta\tilde{u} + (1 + \tilde{\gamma}\widetilde{B})\tilde{u} - \tilde{\gamma}(1 - \tfrac{1}{2}\widetilde{R}_M r^2)\tilde{u}^3, \tag{4.48}$$

where the fields $\widetilde{\beta}_A$, $\tilde{\gamma}$, $\widetilde{B}$, and $\widetilde{R}_M$ are defined by

$$\begin{aligned} \widetilde{\beta}_A(r) &= \frac{r}{r + r_0}\beta_A(r + r_0), \\ \tilde{\gamma}(r) &= \gamma(r + r_0), \\ \widetilde{B}(r) &= \frac{r}{r + r_0}B(r + r_0), \\ \widetilde{R}_M(r)r^2 &= R_M(r + r_0)(r + r_0)^2. \end{aligned} \tag{4.49}$$

Observe that the $\tilde{u}$-equation (4.48) is of the same form as the $u$-equation (3.3), except that the fields $\tilde{\gamma}$ and $\widetilde{B}$ are defined by the relations (4.49) rather than in terms of $\widetilde{\beta}_A$ by the analogues of (3.1) and (3.2). Unlike the scaling transformation (3.27), the geometric meaning of the transformation (4.47) is unclear. However, the global existence Theorem 3.7 uses only the fields $\gamma$ and $B$ and does not require their defining relations, and therefore applies equally well to (4.48).

It follows from the definition of the $C^{(k+\alpha)}$ norm that (for example)

$$\|\tilde{\gamma}\|_{(2+\alpha),[a;b]} \leq \|\gamma\|_{(2+\alpha),[a+r_0;b+r_0]}$$

for any $0 \leq a < b$, and we find therefore

$$\begin{aligned} r^{-1}\widetilde{\beta}_A &\in C^{(3+\alpha)}(A_{[0;\infty)}), \\ \tilde{\gamma} &\in C^{(2+\alpha)}(A_{[0;\infty)}), \\ r^{-1}\widetilde{B} &\in C^{(\alpha)}(A_{[0;\infty)}), \\ \widetilde{R}_M(r)r^2 &\in C^{(\alpha)}(A_{[0;\infty)}). \end{aligned} \tag{4.50}$$

Since $\tilde{\gamma} > 0$ and $\widetilde{R}_M(r)r^2 < 2$, Theorem 3.7 gives a solution $\tilde{u} \in C^{(2+\alpha)}(A_{(0;\infty)})$ to (4.48), bounded by

$$0 < \tilde{\delta}_*(r) \leq \tilde{u}^2(r) \leq \tilde{\delta}^*(r),$$

where $\tilde{\delta}_*(r)$ and $\tilde{\delta}^*(r)$ are defined by (3.18), using $\tilde{\gamma}$, $\widetilde{B}$, and $\widetilde{R}_M$. Now clearly $|\tilde{\gamma}\widetilde{B}(r)| \leq Cr$, so $\tilde{\delta}_*(r)$ and $\tilde{\delta}^*(r)$ can be estimated on $(0;\varepsilon)$, for some small $\varepsilon > 0$, using (4.44):

$$(1-\eta) \leq \tilde{\delta}_*(r) < \tilde{\delta}^*(r) < (1-\eta)^{-1}, \tag{4.51}$$

which translates back to the stated bounds on $u^{-2}$ and $m$, and also shows (for $0 < r < \varepsilon$)

$$\frac{-\eta r}{1-\eta} < 2\tilde{m}(r) < \eta r, \tag{4.52}$$

where $\tilde{m}(r) = \frac{1}{2}r(1-\tilde{u}^{-2}(r)) = m(r+r_0) - \frac{1}{2}r_0$. The rescaling estimate (3.40) applied to $\tilde{m}$ shows that the covariant derivatives of $m$ decay,

$$|\nabla m(r)| + |\nabla^2 m(r)| \leq C(r-r_0),$$

from which it follows as before that the curvature of $g$ is bounded on $A_{[r_0;2r_0]}$.

## 5. Asymptotic decay

The aim of this section is to describe the asymptotic decay of solutions to (3.3) in the special case of vanishing source functions $(\beta_A, R_M)$. The main result shows that the metric approaches the Schwarzschild metric:

**Theorem 5.1.** *Suppose* $r_0 > 0$ *and* $m \in C^\infty(A_{[r_0;\infty)})$ *satisfies*

$$\partial_r m = \tfrac{1}{2} u^{-1} \Delta u, \tag{5.1}$$

*where* $u = (1 - 2m/r)^{-1/2} \in C^\infty(A_{[r_0;\infty)})$. *Then there is a constant* $m_0$ *and* $\varphi \in C^\infty(S^2)$ *satisfying* $(\Delta + 2)\varphi = 0$ *such that*

$$m = m_0 + \frac{\varphi}{r - 2m_0} + O_\infty(r^{-3} \log r), \tag{5.2}$$

*where* $f \in O_k(g(r))$ *means for all* $i, j \geq 0$, $i + j \leq k$, *and* $r \geq r_0$, *there are constants* $C_{i,j}$ *with*

$$|(r\partial_r)^i \nabla^j f| \leq C_{i,j} g(r), \tag{5.3}$$

*and* $O_\infty(g) = \bigcap_{k=1}^\infty O_k(g)$.

The proof of Theorem 5.1 will follow from a series of estimates. We assume throughout this section that $(\beta_A, R_M) \equiv 0$ for all $r \geq r_0$. It will be convenient to define

$$\rho(r, \vartheta) = r - 2m(r, \vartheta), \tag{5.4}$$

$$M(r) = \frac{1}{4\pi} \oint_{S_r} m(r, \vartheta)\, d\sigma, \tag{5.5}$$

and recalling the notation from §2,

$$\begin{aligned} m^*(r) &= \sup\{m(r, \vartheta) : \vartheta \in S^2\}, \\ m_*(r) &= \inf\{m(r, \vartheta) : \vartheta \in S^2\}, \end{aligned} \tag{5.6}$$

we set $\mu_* = m_*(r_0)$ and $\mu^* = m^*(r_0)$. A prime ($'$) will sometimes be used for $d/dr$, and we use the $L^p(S^2)$ norms $\|f(r)\|_p$, $1 \leq p \leq \infty$. Generic constants depending (not depending) on the solution $m(r, \vartheta)$ will be denoted by uppercase $C$ (lowercase $c$).

**Lemma 5.2.** *The following bounds hold:*

$$m_*'(r) \geq 0, \qquad m^{*\prime}(r) \leq 0; \tag{5.7}$$

$$\mu_* \leq m(r, \vartheta) \leq \mu^*; \tag{5.8}$$

$$M'(r) = \frac{1}{8\pi} \oint_{S_r} \frac{|\nabla m|^2}{\rho^2}\, d\sigma \geq 0; \tag{5.9}$$

*and there is a constant* $C_1$, *depending only on* $\mu_*$, $\mu^*$, *and* $r_0$, *such that*

$$\oint_{S_r} (m - M)^2\, d\sigma \leq \frac{C_1}{r^2}. \tag{5.10}$$

*Proof.* There is $\vartheta^* \in S^2$ such that $m(r, \vartheta^*) = m^*(r)$, $\nabla m(r, \vartheta^*) = 0$, and $\Delta m(r, \vartheta^*) \leq 0$, hence $\partial_r m(r, \vartheta^*) \leq 0$ and $m^{*\prime}(r) \leq 0$; similarly $m_*' \leq 0$. Since $u^{-1}\nabla u = \nabla m/\rho$, we have

$$M'(r) = \frac{1}{8\pi}\oint_{S_r} u^{-1}\Delta u = \frac{1}{8\pi}\oint_{S_r} |\nabla m|^2/\rho^2,$$

giving (5.9), and

$$\frac{d}{dr}\oint_{S_r}(m-M)^2 = \oint(mu^{-1}\Delta u - 2MM') = -\oint\frac{|\nabla m|^2}{\rho^2}(\rho + M - m)$$
$$= -\frac{1}{r}\oint|\nabla m|^2\left(1 + \frac{1}{r}\frac{m + M - 4m^2/r}{(1-2m/r)^2}\right).$$

The last term is estimated using (5.8),

$$-\frac{m + M - 4m^2/r}{(1-2m/r)^2} \leq \frac{4\mu^{*2}/r_0 - 2\mu_*}{(1 - 2\mu_*/r_0)} =: C,$$

and the Poincaré inequality gives

$$\oint(m-M)^2 \leq \tfrac{1}{2}\oint|\nabla m|^2,$$

since $\oint(m-M) = 0$. Hence

$$\frac{d}{dr}\oint_{S_r}(m-M)^2 \leq -\frac{1}{r}\left(2 - \frac{C}{r}\right)\oint(m-M)^2,$$

which may be integrated to yield (5.10) with $C_1 = 4\pi e^{C/r_0}(\mu^* - \mu_*)^2$. q.e.d.

There are two invariances of (5.1) which can be used to normalize $m$. The scaling invariance

$$\tilde{m}(r, \vartheta) = \lambda m(r/\lambda, \vartheta), \qquad \lambda > 0,\ r \geq \lambda r_0, \tag{5.11}$$

has been described in §3, while the "translational" invariance

$$\tilde{m}(r, \vartheta) = m(r + \lambda, \vartheta) - \tfrac{1}{2}\lambda, \qquad \lambda > 0,\ r \geq r_0 - \lambda, \tag{5.12}$$

appeared in the proof of Theorem 4.6. In both cases it is readily checked that $\tilde{m}$ satisfies the source-free equation (5.1).

**Lemma 5.3.** *Suppose* $r_1 = \max\{r_0, 30\mu^* - 28\mu_*\}$. *Then there is a constant* $C_2$ *depending only on* $r_1$, $\mu^*$, $\mu_*$, *and* $\|\nabla m(r_1)\|_\infty$ *such that for all* $r \geq r_1$,

$$|\nabla m(r)|^2 \leq C_2/r. \tag{5.13}$$

*Proof.* Let us assume initially that $\frac{1}{2} \leq \mu_* \leq \mu^* \leq 1$; this normalization will later be removed. We use a maximum principle argument with the test function

$$f(r, \vartheta) = |\nabla m|^2 - 2am/r,$$

where $a > 0$ is a constant to be chosen. From (5.1) and the Ricci identity we find

$$\rho \partial_r m = \tfrac{1}{2}\Delta m + \tfrac{3}{2}|\nabla m|^2/\rho^2, \tag{5.14}$$

$$\begin{aligned} \rho \partial_r(|\nabla m|^2) = {} & \frac{1}{2}\Delta|\nabla m|^2 - (|\nabla^2 m|^2 + |\nabla m|^2) \\ & + \frac{2}{\rho}(3m_{|A}m_{|B}m_{|AB} + |\nabla m|^2 \Delta m) + \frac{12}{\rho^2}|\nabla m|^4, \end{aligned} \tag{5.15}$$

and thus $f$ satisfies

$$\begin{aligned} \rho \partial_r f = {} & \frac{1}{2}\Delta f - (|\nabla^2 m|^2 + |\nabla m|^2) + \frac{2am\rho}{r^2} - \frac{3a|\nabla m|^2}{r\rho} \\ & + \frac{2}{\rho}(3m_{|A}m_{|B}m_{|AB} + |\nabla m|^2 \Delta m) + \frac{12}{\rho^2}|\nabla m|^4. \end{aligned} \tag{5.16}$$

Suppose $a$ is chosen sufficiently large that $f(r_1, \cdot) < 0$ for some $r_1 \geq r_0$, and consider the first point $x^* = (r, \vartheta)$, $r \geq r_1$, where $f(r, \vartheta) = 0$. At $x^*$ we have

$$f = 0 \Leftrightarrow |\nabla m|^2 = 2am/r, \tag{5.17}$$

$$\nabla f = 0 \Leftrightarrow m_{|A}m_{|AB} = am_{|B}/r, \tag{5.18}$$

$$\Delta f \leq 0, \tag{5.19}$$

$$\partial_r f \geq 0, \tag{5.20}$$

and thus

$$\begin{aligned} \rho \partial_r f \leq {} & - (|\nabla^2 m|^2 + |\nabla m|^2) + \frac{2am\rho}{r^2} - \frac{3a|\nabla m|^2}{r\rho} \\ & + \frac{2}{\rho}(3m_{|A}m_{|B}m_{|AB} + |\nabla m|^2 \Delta m) + \frac{12}{\rho^2}|\nabla m|^4. \end{aligned} \tag{5.21}$$

We will use (5.17)-(5.19) to obtain a contradiction to (5.20), for $a$ sufficiently large. From (5.17), $\nabla m(x^*) \neq 0$ and we choose the frame $v_1, v_2$ at $x^*$ such that $\nabla m = m_{|1}v_1$ (i.e., $m_{|2} = 0$). Now (5.18) shows

$m_{|11} = a/r$, $m_{|12} = 0$, and thus

$$\rho\partial_r f \leq -\left(\frac{2am}{\rho} + \frac{a^2}{r^2} + m_{|22}^2\right) + \frac{2am\rho}{r^2} - \frac{6a^2 m}{\rho r^2} + \frac{2}{\rho}\left(\frac{6a^2 m}{r^2} + \frac{2am}{r}\left(\frac{a}{r} + m_{|22}\right)\right) + \frac{48a^2 m^2}{r^2\rho^2};$$

using the Schwarz inequality to eliminate $m_{|22}$ and gathering like terms give

$$\rho\partial_r f \leq -\frac{a^2}{r^2}\left(1 - 10\frac{m}{\rho} - 52\frac{m^2}{\rho^2}\right) - \frac{2am}{r^2}\left(r - \frac{\rho}{r}\right). \tag{5.22}$$

The normalization $\frac{1}{2} \leq m \leq 1$ implies $1/(r-2) \leq m/\rho \leq 1/(2r-2)$, so choosing $r \geq 16$ ensures the coefficients of both terms are positive, hence $\partial_r f(x^*) < 0$, contradicting (5.20). Thus setting $r_1 = \max\{r_0, 16\}$ and $a = \|\nabla m(r_1)\|_\infty^2$ yields

$$|\nabla m|^2(r) \leq \frac{2r_1}{r}\|\nabla m(r_1)\|_\infty^2 \quad \forall r \geq r_1, \tag{5.23}$$

for the normalized solution.

Denoting the normalized solution by $\tilde{m}(\tilde{r})$ and the original solution by $m(r)$, we have

$$\tilde{m}(\tilde{r}) = \frac{1}{\lambda}(m(r) - \mu), \tag{5.24}$$

where $r = \lambda\tilde{r} + \mu$. Since we may assume without loss of generality that $\mu_* \neq \mu^*$, choosing $\mu = 2\mu_* - \mu^*$ and $\lambda = 2(\mu^* - \mu_*)$ ensures the normalization $\frac{1}{2} \leq \tilde{m} \leq 1$. The estimate (5.23) translates to

$$|\nabla m|^2(r) \leq \frac{2(r_1 - 2\mu)}{r - 2\mu}\|\nabla m(r_1)\|_\infty^2, \tag{5.25}$$

and the condition $\tilde{r} \geq 16$ will be satisfied if

$$r \geq r_1 := \max\{r_0, 30\mu^* - 28\mu_*\}.$$

**Lemma 5.4.** *There is a constant $C_2$, depending only on $\mu_*$, $\mu^*$, $r_1$, and $\|\nabla m(r_1)\|_\infty^2$, such that for all $r \geq r_1$,*

$$\oint_{S_r} |\nabla m|^2 \leq \frac{C_2}{r^2}. \tag{5.26}$$

*Proof.* Rewriting (5.15) as

$$\begin{aligned}\partial_r|\nabla m|^2 &= -(|\nabla^2 m|^2+|\nabla m|^2)/\rho+\tfrac{1}{2}\Delta(|\nabla m|^2/\rho)\\ &\quad+(2m_{|A}m_{|B}m_{|AB}+|\nabla m|^2\Delta m)/\rho^2+8|\nabla m|^4/\rho^3\\ &= -(|\nabla^2 m|^2+|\nabla m|^2)/\rho+\tfrac{1}{2}\Delta(|\nabla m|^2/\rho)\\ &\quad+\nabla_A(|\nabla m|^2 m_{|A}/\rho^2)/\rho^3+4|\nabla m|^4,\end{aligned}$$

and integrating gives

$$\frac{d}{dr}\oint_{S_r}|\nabla m|^2=-\oint(|\nabla^2 m|^2+|\nabla m|^2)/\rho+4\oint|\nabla m|^4/\rho^4.$$

Using the Ricci identity and then expanding $m$ in spherical harmonics, it is easily verified that

$$\oint(|\nabla^2 m|^2+|\nabla m|^2)=\oint(\Delta m)^2\geq 2\oint|\nabla m|^2,$$

hence applying Lemma 5.3 we see

$$\frac{d}{dr}\oint_{S_r}|\nabla m|^2\leq-\frac{2}{r-2\mu}\left(1-\frac{2C_2}{(r-2\mu_*)^2}\right)\oint|\nabla m|^2,$$

where $\mu$ and $\mu^*$ are as defined above and $r-2m\leq r-2\mu$, since $\mu\leq\mu_*$. Integrating this inequality yields the required bound. q.e.d.

The decay bounds of Lemmas 5.2 and 5.3 are the key to applying a general technique for showing decay of all derivatives for solutions to a parabolic equation ([17], [13]). (The exposition here follows the model of [13], and the author is indebted to Piotr Chruściel for discussions on this topic.)

**Lemma 5.5.** *For each $k\in\mathbf{Z}^+$, there is a constant $C_3=C_3(k)$, depending also on $C_2$, $\mu_*$, $\mu^*$, and $\|\nabla m(r_1)\|_2$, such that*

$$\oint_{S_r}|\nabla^k m|^2\leq\frac{C_3(k)}{r^2}\tag{5.27}$$

*for all $r\geq 2r_1$.*

*Proof.* Let $v_1$, $v_2$, $v_3$ be the symmetry generating vector fields on $S^2$, normalized by the commutation relations $[v_1, v_2]=v_3$ and cyclic permutations. Regarding the $v_i$, $i=1,\cdots,3$, as differential operators on $S^2$, we have

$$\sum_{i=1}^{3}(v_i)^2=\Delta=-D^2,\tag{5.28}$$

where $D = (-\Delta)^{-1/2}$ is defined spectrally. The following facts are easily verified:

$$[v_i, \Delta] = 0, \tag{5.29}$$

$$\oint v_i(f) = 0 \quad \text{for all } f \in C^1(S^2), \tag{5.30}$$

$$\sum_{|I|=j} \oint (v_I f)^2 = \begin{cases} \oint (\Delta^n f)^2 & \text{if } j = 2n, \\ \oint |\nabla \Delta^n f|^2 & \text{if } j = 2n+1, \end{cases} \tag{5.31}$$

where $I = (i_1, \cdots, i_j)$ is a multi-index, $i_1, \cdots, i_j = 1, 2, 3$, and $v_I = v_{i_1} v_{i_2} \cdots v_{i_j}$.

It will suffice to instead estimate

$$E_k(r) := \oint |D^k m|^2,$$

since there is a constant $c = c(k)$ such that for all $f \in C^\infty(S^2)$,

$$\oint |\nabla^k f|^2 \leq c \oint |D^k f|^2. \tag{5.32}$$

This is standard and can be shown directly: by repeated integration by parts and the Ricci identity, we have (for $k$ even; $k$ odd is treated similarly)

$$\oint |\nabla^k f|^2 = \oint |\Delta^{k/2} f|^2 + \oint |P_{k-1} f|^2$$

for some homogeneous differential operator $P_{k-1}$ of order $k-1$. Applying (5.31) and induction gives (5.32).

We use the shorthand notation $\sum_j^* v_I f$ to denote a generic linear combination of terms $v_I f$, $|I| = j$, with coefficients depending on $k$ and perhaps on other parameters, but independent of $f$. Furthermore, let $J_s = J_s(r)$ denote the generic term of the form

$$J_s \sim \oint v_{I_1}(m) \cdots v_{I_s}(m) \rho^{1-s},$$

where $s, I_1, \cdots, I_s$ satisfy $s \geq 3$, $|I_j| \leq k$ for $1 \leq j \leq s$, and $\sum_{j=1}^s |I_j| \leq 2k+2$, and set $J = \sum_{s\geq 3} J_s$.

Let $I$, $|I| = k \geq 2$, be any multi-index; then from the basic equation (5.1) and the commutation relations we have

$$\frac{d}{dr} \oint_{S_r} |v_I m|^2 = \oint v_I(m) \cdot (v_I(\Delta m/\rho) + 3 v_I(|\nabla m|^2/\rho)). \tag{5.33}$$

Commuting derivatives,

$$\oint v_I(m)v_I(\Delta m/\rho) = \oint \frac{1}{\rho}v_I(m)\Delta(v_I m) + \oint v_I(m)\sum_{j=1}^{k}\frac{2}{\rho^2}v_{i_j}(m)\Delta v_{I'_j}(m) + \sum_{s=3}^{k+2} J_s, \tag{5.34}$$

where $I'_j = (i_1, \cdots, \hat{i}_j, \cdots, i_k)$ (here $\hat{i}_j$ denotes an omitted term). The first term on the right of (5.34) is handled by integration by parts,

$$\begin{aligned}\oint v_I(m)v_I\left(\frac{\Delta m}{\rho}\right) &= -\sum_{j=1}^{3}\oint\frac{1}{\rho}|v_j v_I m|^2 - \oint\frac{1}{\rho^2}v_j(m)v_j(|v_I m|^2)\\ &= -\oint\frac{1}{\rho}|Dv_I m|^2 + \oint\frac{1}{\rho^3}|v_I m|^2(\rho\Delta m + 4|\nabla m|^2)\\ &= -\oint\frac{1}{\rho}|Dv_I m|^2 + J_3 + J_4.\end{aligned}$$

From the commutation relations we have

$$v_I(m) = v_{I_j}v_{I'_j}m + \sum_{|I''|=k-1}^{*} v_{I''}m,$$

and the second term of (5.34) is estimated by repeated integration by parts,

$$\begin{aligned}&\oint\frac{1}{\rho^2}v_I(m)v_i(m)\Delta v_{I'}(m)\\ &= \oint\frac{1}{\rho^2}\left(v_i v_{I'}(m) + \sum_{|I''|=k-1}^{*} v_{I''}(m)\right)v_j v_j v_{I'}(m)v_i(m)\\ &= -\tfrac{1}{2}\oint\frac{1}{\rho^2}v_i(|v_j v_{I'}(m)|^2)v_i(m) + J_3 + J_4\\ &= J_3 + J_4,\end{aligned}$$

showing that (5.34) becomes

$$\oint v_I(m)v_I\left(\frac{\Delta m}{\rho}\right) = -\oint|Dv_I m|^2/\rho^2 + J.$$

The second term of (5.33) is handled similarly, giving finally that

$$\frac{d}{dr}\oint|D^k m|^2 \le -\oint|D^{k+1}m|^2 + J. \tag{5.35}$$

Hölders inequality with $\sum_1^s 1/p_j = 1$ implies

$$J_s \le \frac{c}{(r-2\mu^*)^{s-1}}\|v_{I_1}m\|_{p_1}\cdots\|v_{I_s}m\|_{p_s}, \tag{5.36}$$

where $|I_j| \le k$, $\sum |I_j| = 2k+2$, and $s \ge 3$. The directional derivatives $v_I$ can be expanded in terms of covariant derivatives, hence for any $f \in C^\infty(S^2)$ by interpolating we obtain

$$\|v_I f\|_p \le c(\|\nabla^j f\|_p + \|f\|_p), \qquad j = |I|.$$

The Gagliardo-Nirenberg inequality states

$$\|\nabla^j f\|_p \le c\|\nabla^k f\|_2^a \|f\|_q^{1-a},$$

with $a = (j + 2/q - 2/p)/(k - 1 + 2/q)$, $1 \le j < k$, $1 < p, q < \infty$. From (5.32) with $f = v_i(m)$ it follows that

$$\|v_{I_j} m\|_p \le c(\|D^{k+1} m\|_2^{a_j} \|\nabla m\|_{q_j}^{1-a_j} + \|\nabla m\|_{p_j}),$$

with $a_j = (|I_i| - 1 + 2/q_j + 2/p_j)/(k - 1 + 2/q_j)$. Choosing $p_j = s = q_j$ and expanding out the product in (5.36) give

$$J_s \le \frac{c}{(r - 2\mu^*)^{s-1}} \sum_{j=1}^{s} \|D^{k+1} m\|_2^{2-\delta_j} \|\nabla m\|_s^{s+\delta_j-2},$$

for some $\delta \le \delta_j \le 2$, where $\delta = 2 - \sum a_j = (s + 4/s - 4)/(k - 1 + 2/s) > 0$ for $s \ge 3$. Now use of the Young's inequality leads to

$$J_s \le \frac{1}{(r - 2\mu^*)^{s-1}} (\varepsilon \|D^{k+1} m\|^2 + F_s(\varepsilon, \|\nabla m\|_s))$$

for some smooth function $F_s$ satisfying

$$F_s(\varepsilon, x) \le c(\varepsilon) x^s \quad \text{for } 0 \le x \le 1.$$

The $L^2$ and $L^\infty$ bounds for $\nabla m$ show that

$$F_s(\varepsilon, \|\nabla m\|_s) \le C r^{-(1+s/2)}$$

for some constant $C$ depending on $\varepsilon$, $r_0$, $r_1$, $\mu_*$, $\mu^*$, and $\|\nabla m(r_0)\|_\infty$. Now choosing $\varepsilon$ appropriately we finally obtain

$$\text{(5.37)} \qquad \frac{d}{dr} \oint |D^k m|^2 \le - \oint |D^{k+1} m|^2 \left( \frac{1}{\rho} - \frac{\varepsilon}{(r - 2\mu^*)^2} \right) + C(\varepsilon) r^{-9/2}.$$

Assuming for convenience the normalization $\frac{1}{2} \le \mu_* \le \mu^* \le 1$, for all $k \ge 1$ we have

$$\text{(5.38)} \qquad \frac{d}{dr} E_k(r) \le -\frac{1}{r} E_{k+1}(r) + C_k r^{-9/2}.$$

Following [17] and [13], we define

$$E(r) = \sum_{j=1}^{k} R_j(r) E_j(r),$$

where we choose

$$R_j(r) = \frac{1}{(j-1)!}\left(\log\frac{r}{r_2}\right)^{j-1}$$

for $r_1 \le r_2 \le r$. Then

$$\begin{aligned}\frac{d}{dr}E(r) &\le \sum_2^k \left(R_j' - \frac{1}{r}R_{j-1}\right)E_j + \sum_1^k r^{-9/2}R_jC_j \\ &\le \sum_0^{k-1}\frac{C_{j+1}}{j!}\left(\log\frac{r}{r_2}\right)^j r^{-9/2}\end{aligned}$$

for $r \ge r_2$. Noting that $E(r_2) = \oint |\nabla m|^2(r_2)$, by integrating from $r_2$ to $2r_2$ we obtain

$$E(2r_2) \le C_k r_2^{-3} + \oint_{S_{r_2}} |\nabla m|^2,$$

and in particular applying Lemma 5.4 with $r = 2r_2 \ge 2r_1$,

$$\oint_{S_r} |D^k m|^2 = E_k(r) \le \frac{C_k}{r^2},$$

which concludes the proof. q.e.d.

The proof of Theorem 5.1 now follows readily. From (5.27) and the Sobolev inequality it follows that

$$|\nabla^k m| \le C_k/r \tag{5.39}$$

for all $k \ge 1$, $r \ge 2r_1$ (where $C_k$ depends only on $\varepsilon$, $r_0$, $r_1$, $\mu_*$, $\mu^*$, and $\|\nabla m(r_0)\|_\infty$), hence there is $m_0 \in \mathbf{R}$ such that

$$|m - m_0| \le C/r \quad \text{for } r \ge 2r_1. \tag{5.40}$$

The decay estimates (5.40) imply that $\tilde{m} := m - m_0$ satisfies

$$2(r - 2m_0)\partial_r\tilde{m} = \Delta\tilde{m} + f, \tag{5.41}$$

where $f = O_\infty(r^{-3})$. The eigenfunctions (spherical harmonics) $\varphi_{l,j}$, $l = 0, 1, \cdots, j = -l, -l+1, \cdots, l$, form a complete orthonormal basis for $L^2(S^2)$, with eigenvalues

$$\Delta\varphi_{l,j} = -l(l+1)\varphi_{l,j},$$

and the Fourier coefficients

$$F_{l,j}(r) = \oint_{S_r} f\varphi_{l,j}$$

therefore satisfy $|F_{l,j}| \le C/r^3$. Since $f$ is smooth, writing $\varphi_{l,j} = (-1)^N l^{-N}(l+1)^{-N}\Delta^N \varphi_{l,j}$ and repeated integration by parts show there is a constant $K_N$ for any $N \in \mathbf{Z}^+$, such that for all $r \ge 2r_1$,

$$|F_{l,j}(r)| \le K_N l^{-2N} r^{-3}.$$

Defining also

$$M_{l,j}(r) = \oint_{S_r} m\varphi_{l,j},$$

we see from (5.41) that for $l \ge 1$,

$$2(r-2m_0)\frac{d}{dr}M_{l,j} = -l(l+1)M_{l,j} + F_{l,j}.$$

Since $m(2r_1) \in C^\infty(S^2)$, there are constants $C_N$ such that $|M_{l,j}(2r_1)| \le C_N l^{-2N}$ $\forall N$, $l \ge 1$, and hence

$$M_{1,j} = \frac{k_{1,j}}{r-2m_0} + O(r^{-3})$$

for some constants $k_{1,j}$, and in general,

$$|M_{2,j}| \le Cr^{-3}\log r,$$
$$|M_{l,j}| \le C_N K_N l^{-2N} r^{-3} \quad \forall l \ge 3,\ N \ge 1.$$

Thus $\sum_{l\ge 3} M_{l,j}\varphi_{l,j} = O_\infty(r^{-3})$, and letting $\varphi = \sum k_{1,j}\varphi_{1,j}$ we have

$$m = m_0 + \frac{\varphi}{r-2m_0} + O_\infty(r^{-3}\log r),$$

where $\varphi$ is the required first eigenfunction of $\Delta$.

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